Perturbations of stars and black holes

Valeria Ferrari

Universita' di Roma "La Sapienza"

How to get approximate estimates of the amount of GW radiation emitted by a dynamically evolving system:

THE QUADRUPOLE FORMALISM

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| << 1$$

Assumptions:

- the gravitational field is assumed to be weak, so that gravitational interactions do not dominate
- the velocities of the bodies involved in the problem are much smaller than the speed of light

$v << c \quad \rightarrow \quad \Omega R << c \quad$ R= source extension

$$\Downarrow$$

which implies that the region where the source is confined is small compared to the wavelenght of the emitted radiation $\lambda_{GW} = \frac{2\pi c}{\Omega}$

$$\begin{cases} h^{ij} = \dfrac{2G}{c^4}\dfrac{e^{i\omega\frac{r}{c}}}{r}\left[\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\; q^{ij}\right] \\ h^{\mu 0} = 0, \quad \mu = 0,3 \end{cases}, \qquad q^{ij} = \int_V T^{00} x^i x^j dV$$

where q^{ij} is the **quadrupole moment** of the source, which depends only on the **time variation of the energy density** of the source.

In spite of the very strong assumptions this formula gives good results for

- emission by triaxial neutron stars
- evolution of binary systems before they get too close

For instance, for the binary system PSR 1931+16 (Taylor-pulsar)

$$M_1 \sim M_2 \sim 1.4 M_\odot, \qquad \mathbf{D \sim 2R_\odot = 13.8 \cdot 10^{10}\ cm}$$
$$T = 7h\ 45m\ 7s \qquad \nu_{GW} \sim 1,4 \cdot 10^{-4}\ Hz$$

$$\boldsymbol{\lambda_{GW} = \frac{c}{\nu_{GW}} \sim 10^{14}\ cm} \qquad \boldsymbol{\lambda_{GW} >> D}$$

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$$T = 7h\ 45m\ 7s \qquad \nu_{GW} \sim 1,4 \cdot 10^{-4}\ Hz$$

$$\lambda_{\mathbf{GW}} = \frac{\mathbf{c}}{\nu_{\mathbf{GW}}} \sim \mathbf{10^{14}\ cm} \qquad \lambda_{\mathbf{GW}} >> \mathbf{D}$$

But if we want to describe a pulsating neutron star

$$\lambda_{\mathbf{GW}} = \frac{c}{\nu_{GW}} \sim \frac{3 \cdot 10^{10}\ cm/s}{2 - 3 \cdot 10^3\ Hz} \sim \mathbf{10^7\ cm}$$
$$\mathbf{D \sim 20\ km = 2 \cdot 10^6\ cm}$$

the quadrupole formalism cannot be applied

In addition

- stars and black holes scatter incident radiation, and the scattering can be resonant
- the spacetime is curved and it has a dynamical behaviour

how to take these phenomena into account?

$$g_{\mu\nu} = g^{\mathbf{0}}_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| << |g^{\mathbf{0}}_{\mu\nu}|$$

$g^{\mathbf{0}}_{\mu\nu}$ is an exact sol. of Eintsein's eqs.

T.Regge, J.A. Wheeler *Phys. Rev.*108 1063, 1957

Initial motivation: the study of stability

"A sphere of water held together by gravitational forces is stable against small departures from sphericity. A sphere of water surrounded by a spherical shell of liquid mercury is also an equilibrium configuration for gravitational forces, but a situation of unstable equilibrium. Initial small departures from sphericity at the water-mercury interface will grow exponentially, and the mercury will concentrate with a rush at the center of the sphere. Which situation will more closely correspond to the behavior of a Schwarzschild singularity subjected to a small initial perturbation? "

- **1957** equations for the axial perturbations of a Schwarzschild black hole
- **1970** equations for the polar perturbations + source term (Zerilli)
- **1970** Vishveshwara introduces the notion of **quasi-normal modes of black holes**
- **1975** Chandrasekhar and Detweiler actually compute the discrete **eigenfrequencies** of the quasi-normal modes of a Schwarzschild black hole.

The theory of black hole perturbations focuses mainly on two problems:

- calculation of the eigenfrequencies of the quasi-normal modes for rotating and non-rotating black holes
- solution of the inhomogeneous equations to compute waveforms and energy emitted when the perturbation is excited by some specific source.

Tensor Harmonics

A suitable basis to expand symmetric tensors is formed by the following tensor harmonics
(Regge, J.A. Wheeler, 1957, F. Zerilli, F. Zerilli, 1970),

$$
\begin{aligned}
\mathbf{a}_{\ell m} &= [\mathbf{e}_r \mathbf{e}_r \; Y_{\ell m}] \\
\mathbf{b}_{\ell m} &= 2^{\frac{1}{2}} n(\ell) r \, [\mathbf{e}_r \nabla \; Y_{\ell m}] \\
\mathbf{c}_{\ell m} &= 2^{\frac{1}{2}} n(\ell) \, [\mathbf{e}_r \mathbf{L} \; Y_{\ell m}] \\
\mathbf{d}_{\ell m} &= 2^{\frac{1}{2}} m(\ell) r \left\{ [\mathbf{L} \nabla \; Y_{\ell m}] + \frac{1}{r} [\mathbf{e}_r \mathbf{L} \; Y_{\ell m}] \right\} \\
\mathbf{f}_{\ell m} &= 2^{-\frac{1}{2}} m(\ell) \, (\mathbf{e}_{\ell m} + \mathbf{h}_{\ell m}) \\
\mathbf{g}_{\ell m} &= -2^{-\frac{1}{2}} n(\ell)^2 \, (\mathbf{e}_{\ell m} - \mathbf{h}_{\ell m}) \\
\mathbf{a}^{(0)}_{\ell m} &= [\mathbf{e}_t \mathbf{e}_t \; Y_{\ell m}] \\
\mathbf{a}^{(1)}_{\ell m} &= 2^{\frac{1}{2}} \, [\mathbf{e}_t \mathbf{e}_r \; Y_{\ell m}] \\
\mathbf{b}^{(0)}_{\ell m} &= 2^{\frac{1}{2}} n(\ell) r \, [\mathbf{e}_t \nabla \; Y_{\ell m}] \\
\mathbf{c}^{(0)}_{\ell m} &= 2^{\frac{1}{2}} n(\ell) \, [\mathbf{e}_t \mathbf{L} \; Y_{\ell m}]
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{e}_{\ell m} &= r^2 \left\{ [\nabla \nabla \; Y_{\ell m}] + \frac{2}{r} [\mathbf{e}_r \nabla \; Y_{\ell m}] \right\} \\
\mathbf{h}_{\ell m} &= [\mathbf{L}\mathbf{L} \; Y_{\ell m}] + r \, [\mathbf{e}_r \nabla \; Y_{\ell m}] \, ,
\end{aligned}
$$

and

$$
\begin{aligned}
n(\ell) &= [\ell(\ell+1)]^{-\frac{1}{2}} \\
m(\ell) &= [\ell(\ell+1)(\ell-1)(\ell+2)]^{-\frac{1}{2}} \, .
\end{aligned}
$$

Any symmetric tensor can be expanded as follows

$$
\begin{aligned}
\mathbf{T} = \sum_{\ell m} \Big[& A^{(0)}_{\ell m} \mathbf{a}^{(0)}_{\ell m} + A^{(1)}_{\ell m} \mathbf{a}^{(1)}_{\ell m} + A_{\ell m} \mathbf{a}_{\ell m} + \\
& + B^{(0)}_{\ell m} \mathbf{b}^{(0)}_{\ell m} + B_{\ell m} \mathbf{b}_{\ell m} + Q^{(0)}_{\ell m} \mathbf{c}^{(0)}_{\ell m} + \\
& + Q_{\ell m} \mathbf{c}_{\ell m} + G_{\ell m} \mathbf{g}_{\ell m} + D_{\ell m} \mathbf{d}_{\ell m} + F_{\ell m} \mathbf{f}_{\ell m} \Big] \, ,
\end{aligned}
$$

and the coefficients of the expansion are given by

$$
A^{(0)}_{\ell m} = \int \mathbf{a}^{(0)\mu\nu *}_{\ell m} \; T_{\mu\nu} d\Omega
$$

The explicit expressions of the tensors

$$
\mathbf{a}^{(0)}_{\ell m} = \begin{pmatrix} (t) & (\varphi) & (r) & (\vartheta) \\ Y_{\ell m}(\varphi,\vartheta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{a}^{(1)}_{\ell m} = \frac{1}{\sqrt{2}} \begin{pmatrix} (t) & (\varphi) & (r) & (\vartheta) \\ 0 & 0 & Y_{\ell m}(\varphi,\vartheta) & 0 \\ 0 & 0 & 0 & 0 \\ Y_{\ell m}(\varphi,\vartheta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
\mathbf{a}_{\ell m} = \begin{pmatrix} (t) & (\varphi) & (r) & (\vartheta) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Y_{\ell m}(\varphi,\vartheta) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{b}^{(0)}_{\ell m} = \frac{n(\ell) r}{\sqrt{2}} \begin{pmatrix} (t) & (\varphi) & (r) & (\vartheta) \\ 0 & \frac{\partial Y_{\ell m}}{\partial \varphi} & 0 & \frac{\partial Y_{\ell m}}{\partial \vartheta} \\ \frac{\partial Y_{\ell m}}{\partial \varphi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\partial Y_{\ell m}}{\partial \vartheta} & 0 & 0 & 0 \end{pmatrix}
$$

$$
\mathbf{b}_{\ell m} = \frac{n(\ell) r}{\sqrt{2}} \begin{pmatrix} (t) & (\varphi) & (r) & (\vartheta) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial Y_{\ell m}}{\partial \varphi} & 0 \\ 0 & \frac{\partial Y_{\ell m}}{\partial \varphi} & 0 & \frac{\partial Y_{\ell m}}{\partial \vartheta} \\ 0 & 0 & \frac{\partial Y_{\ell m}}{\partial \vartheta} & 0 \end{pmatrix}, \quad \mathbf{f}_{\ell m} = \frac{m(\ell) r^2}{\sqrt{2}} \begin{pmatrix} (t) & (\varphi) & (r) & (\vartheta) \\ 0 & 0 & 0 & 0 \\ 0 & -\sin^2\vartheta W_{\ell m} & 0 & X_{\ell m} \\ 0 & 0 & 0 & 0 \\ 0 & X_{\ell m} & 0 & W_{\ell m} \end{pmatrix}
$$

$$
\mathbf{g}_{\ell m} = \frac{r^2}{\sqrt{2}} \begin{pmatrix} (t) & (\varphi) & (r) & (\vartheta) \\ 0 & 0 & 0 & 0 \\ 0 & \sin^2\vartheta Y_{\ell m} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_{\ell m} \end{pmatrix}
$$

$$
\mathbf{c}^{(0)}_{\ell m} = \frac{\imath n(\ell) r}{\sqrt{2}} \begin{pmatrix} (t) & (\varphi) & (r) & (\vartheta) \\ 0 & -\sin\vartheta \frac{\partial Y_{\ell m}}{\partial \vartheta} & 0 & \frac{1}{\sin\vartheta}\frac{\partial Y_{\ell m}}{\partial \varphi} \\ -\sin\vartheta \frac{\partial Y_{\ell m}}{\partial \vartheta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sin\vartheta}\frac{\partial Y_{\ell m}}{\partial \varphi} & 0 & 0 & 0 \end{pmatrix}
$$

$$
\mathbf{c}_{\ell m} = \frac{\imath n(\ell) r}{\sqrt{2}} \begin{pmatrix} (t) & (\varphi) & (r) & (\vartheta) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\sin\vartheta \frac{\partial Y_{\ell m}}{\partial \vartheta} & 0 \\ 0 & -\sin\vartheta \frac{\partial Y_{\ell m}}{\partial \vartheta} & 0 & \frac{1}{\sin\vartheta}\frac{\partial Y_{\ell m}}{\partial \varphi} \\ 0 & 0 & \frac{1}{\sin\vartheta}\frac{\partial Y_{\ell m}}{\partial \varphi} & 0 \end{pmatrix}
$$

$$
\mathbf{d}_{\ell m} = \frac{\imath m(\ell) r^2}{\sqrt{2}} \begin{pmatrix} (t) & (\varphi) & (r) & (\vartheta) \\ 0 & 0 & 0 & 0 \\ 0 & -\sin\vartheta X_{\ell m} & 0 & -\sin\vartheta W_{\ell m} \\ 0 & 0 & 0 & 0 \\ 0 & -\sin\vartheta W_{\ell m} & 0 & \frac{1}{\sin\vartheta} X_{\ell m} \end{pmatrix}
$$

$$
X_{\ell m}(\vartheta,\varphi) = 2\frac{\partial}{\partial\varphi}\left[\frac{\partial}{\partial\vartheta} - \cot\vartheta\right] Y_{\ell m}(\vartheta,\varphi), \qquad W_{\ell m}(\vartheta,\varphi) = \left[\frac{\partial^2}{\partial^2\vartheta} - \cot\vartheta\frac{\partial}{\partial\vartheta} - \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial^2\varphi}\right] Y_{\ell m}(\vartheta,\varphi).
$$

PARITY

If we apply the parity operator which transforms

$$\theta \to \pi - \theta \qquad \text{and} \qquad \varphi \to \pi + \varphi$$

$\mathbf{c}_{\ell m}, \quad \mathbf{d}_{\ell m}, \quad \mathbf{c}^{(0)}_{\ell m}$ transform like $\quad (-1)^{(\ell+1)}$
and are said **axial** or **odd**

$\mathbf{a}_{\ell m}, \mathbf{b}_{\ell m}, \mathbf{f}_{\ell m}, \mathbf{g}_{\ell m}, \mathbf{a}^{(0)}_{\ell m}, \mathbf{a}^{(1)}_{\ell m}, \mathbf{b}^{(0)}_{\ell m}$ transform like $\quad (-1)^{(\ell)}$
and are said **polar** or **even**

Consequently

$$\mathbf{T}^{\mathbf{ax}} = \sum_{\ell m} \left[Q^{(0)}_{\ell m} \mathbf{c}^{(0)}_{\ell m} + \; + Q_{\ell m} \mathbf{c}_{\ell m} + D_{\ell m} \mathbf{d}_{\ell m} \right].$$

and

$$\begin{aligned} \mathbf{T}^{pol} \; &= \; \sum_{\ell m} \Big[A^{(0)}_{\ell m} \mathbf{a}^{(0)}_{\ell m} + A^{(1)}_{\ell m} \mathbf{a}^{(1)}_{\ell m} + A_{\ell m} \mathbf{a}_{\ell m} + \\ & \quad + B^{(0)}_{\ell m} \mathbf{b}^{(0)}_{\ell m} + B_{\ell m} \mathbf{b}_{\ell m} + G_{\ell m} \mathbf{g}_{\ell m} + F_{\ell m} \mathbf{f}_{\ell m} \Big], \end{aligned}$$

Perturbations of a Schwarzschild black hole

- expand $\mathbf{h}_{\mu\nu}$ in tensorial spherical harmonics

$$\mathbf{h}_{\mu\nu} = \mathbf{h}^{ax}_{\mu\nu\ \ell m} + \mathbf{h}^{pol}_{\mu\nu\ \ell m}$$

- choose a gauge to eliminate **4** metric components

$$\mathbf{h}^{ax}_{\ell m} = \begin{pmatrix} (t) & (\varphi) & (r) & (\vartheta) \\ 0 & \mathbf{h}^{ax}_0 \sin\vartheta \frac{\partial Y_{\ell m}}{\partial\vartheta} & 0 & -\mathbf{h}^{ax}_0 \frac{1}{\sin\vartheta}\frac{\partial Y_{\ell m}}{\partial\varphi} \\ \mathbf{h}^{ax}_0 \sin\vartheta \frac{\partial Y_{\ell m}}{\partial\vartheta} & 0 & \mathbf{h}^{ax}_1 \sin\vartheta \frac{\partial Y_{\ell m}}{\partial\vartheta} & 0 \\ 0 & \mathbf{h}^{ax}_1 \sin\vartheta \frac{\partial Y_{\ell m}}{\partial\vartheta} & 0 & -\mathbf{h}^{ax}_1 \frac{1}{\sin\vartheta}\frac{\partial Y_{\ell m}}{\partial\varphi} \\ -\mathbf{h}^{ax}_0 \frac{1}{\sin\vartheta}\frac{\partial Y_{\ell m}}{\partial\varphi} & 0 & -\mathbf{h}^{ax}_1 \frac{1}{\sin\vartheta}\frac{\partial Y_{\ell m}}{\partial\varphi} & 0 \end{pmatrix}$$

$$\mathbf{h}^{pol}_{\ell m} = \begin{pmatrix} (t) & (\varphi) & (r) & (\vartheta) \\ 2e^{2\nu}\mathbf{N}Y_{\ell m} & 0 & 0 & 0 \\ 0 & -2e^{2\psi}\mathbf{H_{11}} & 0 & -\mathbf{r^2V}X_{\ell m} \\ 0 & 0 & -2e^{2\mu_2}\mathbf{L}Y_{\ell m} & 0 \\ 0 & -\mathbf{r^2V}X_{\ell m} & 0 & -2e^{2\mu_3}\mathbf{H_{33}} \end{pmatrix}$$

with

$$\begin{aligned} H_{11} &= \left[\mathbf{T}Y_{\ell m} + \mathbf{V}\left(\frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} + \cot\vartheta\frac{\partial}{\partial\vartheta}\right) Y_{\ell m}\right] \\ H_{33} &= \left[\mathbf{T}Y_{\ell m} + \mathbf{V}\frac{\partial^2}{\partial\vartheta^2} Y_{\ell m}\right]. \end{aligned}$$

$Y_{\ell m}(\vartheta,\varphi)$ = scalar spherical harmonics

all **blue** quantities are funtions of **t and r**

$\mathbf{h}^{ax}_{\ell m}$ and $\mathbf{h}^{pol}_{\ell m}$ mean that when $\boldsymbol{\vartheta} \to \boldsymbol{\Pi} - \boldsymbol{\vartheta},\ \boldsymbol{\varphi} \to \boldsymbol{\Pi} + \boldsymbol{\varphi}$

$\mathbf{h}^{ax} \to \mathbf{h}^{ax}(-\mathbf{1})^{\boldsymbol{\ell}+1}$ and $\mathbf{h}^{pol} \to \mathbf{h}^{pol}(-\mathbf{1})^{\boldsymbol{\ell}}$

$$g_{\mu\nu} = g^{\mathbf{0}}_{\mu\nu} + h_{\mu\nu}$$

The unperturbed metric $g^{\mathbf{0}}_{\mu\nu}$ is the Schwarzschild metric

$$ds^2 = e^{2\nu(r)}dt^2 - e^{2\mu_2(r)}dr^2 - r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right), \quad e^{2\nu} = e^{-2\mu_2} = \left(1 - \frac{2M}{r}\right)$$

- insert $\mathbf{g}_{\mu\nu}$ in the Einstein equations, with $\mathbf{h}_{\mu\nu}$ expanded in harmonics
- separate the angular part
- Fourier-expand the perturbed functions $f(t) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega t}d\omega$

The axial equations

$$-i\omega e^{-2\nu}\mathbf{h_0} - e^{-2\mu_2}\left[\mathbf{h_{1,r}} + (\nu - \mu_2)_{,r}\mathbf{h_1}\right] = 0$$

$$e^{-2\nu}\left[-i\omega\left(\mathbf{h_{0,r}} - \frac{2}{r}\mathbf{h_0}\right) + \omega^2\mathbf{h_1}\right] - \frac{2n}{r^2}\mathbf{h_1} = 0\,.$$

The Polar equations

$$(\mathbf{T} - \mathbf{V} + \mathbf{N})_{,r} - \left(\frac{1}{r} - \nu_{,r}\right)\mathbf{N} - \left(\frac{1}{r} + \nu_{,r}\right)\mathbf{L} = 0,$$

$$\mathbf{V_{,r,r}} + \left(\frac{2}{r} + \nu_{,r} - \mu_{2,r}\right)\mathbf{V_{,r}} + \frac{e^{2\mu_2}}{r^2}(\mathbf{N} + \mathbf{L}) + \omega^2 e^{2\mu_2 - 2\nu}\mathbf{V} = 0,$$

$$(\mathbf{T} - \mathbf{V} + \mathbf{L}) = 0$$

$$\left[\frac{d}{dr} + \left(\frac{1}{r} - \nu_{,r}\right)\right](2\mathbf{T} - k\mathbf{V}) - \frac{2}{r}\mathbf{L} = 0$$

$$\frac{1}{2}e^{-2\mu_2}\left[\frac{2}{r}\mathbf{N_{,r}} + \left(\frac{1}{r} + \nu_{,r}\right)(2\mathbf{T} - k\mathbf{V})_{,r} - \frac{2}{r}\left(\frac{1}{r} + 2\nu_{,r}\right)\mathbf{L}\right] +$$

$$\frac{1}{2}\left[-\frac{1}{r^2}(2n\mathbf{T} + k\mathbf{N}) + \omega^2 e^{-2\nu}(2\mathbf{T} - k\mathbf{V})\right] = 0$$

$$k = \ell(\ell + 1), \qquad 2n = k - 2$$

$$g_{\mu\nu} = g^{0}_{\mu\nu} + h_{\mu\nu}$$

The unperturbed metric $g^{0}_{\mu\nu}$ is the Schwarzschild metric

$$ds^2 = e^{2\nu(r)}dt^2 - e^{2\mu_2(r)}dr^2 - r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right), \quad e^{2\nu} = e^{-2\mu_2} = \left(1 - \frac{2M}{r}\right)$$

- insert $\mathbf{g}_{\mu\nu}$ in the Einstein equations, with $\mathbf{h}_{\mu\nu}$ expanded in harmonics
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The axial equations

$$-i\omega e^{-2\nu}\mathbf{h_0} - e^{-2\mu_2}\left[\mathbf{h_{1,r}} + (\nu - \mu_2)_{,r}\mathbf{h_1}\right] = 0$$
$$e^{-2\nu}\left[-i\omega\left(\mathbf{h_{0,r}} - \frac{2}{r}\mathbf{h_0}\right) + \omega^2\mathbf{h_1}\right] - \frac{2n}{r^2}\mathbf{h_1} = 0\,.$$

The Polar equations

$$(\mathbf{T} - \mathbf{V} + \mathbf{N})_{,r} - \left(\frac{1}{r} - \nu_{,r}\right)\mathbf{N} - \left(\frac{1}{r} + \nu_{,r}\right)\mathbf{L} = 0,$$
$$\mathbf{V_{,r,r}} + \left(\frac{2}{r} + \nu_{,r} - \mu_{2,r}\right)\mathbf{V_{,r}} + \frac{e^{2\mu_2}}{r^2}(\mathbf{N} + \mathbf{L}) + \omega^2 e^{2\mu_2 - 2\nu}\mathbf{V} = 0,$$
$$(\mathbf{T} - \mathbf{V} + \mathbf{L}) = 0$$
$$\left[\frac{d}{dr} + \left(\frac{1}{r} - \nu_{,r}\right)\right](2\mathbf{T} - k\mathbf{V}) - \frac{2}{r}\mathbf{L} = 0$$
$$\frac{1}{2}e^{-2\mu_2}\left[\frac{2}{r}\mathbf{N_{,r}} + \left(\frac{1}{r} + \nu_{,r}\right)(2\mathbf{T} - k\mathbf{V})_{,r} - \frac{2}{r}\left(\frac{1}{r} + 2\nu_{,r}\right)\mathbf{L}\right] +$$
$$\frac{1}{2}\left[-\frac{1}{r^2}(2n\mathbf{T} + k\mathbf{N}) + \omega^2 e^{-2\nu}(2\mathbf{T} - k\mathbf{V})\right] = 0$$
$$k = \ell(\ell + 1), \qquad 2n = k - 2$$

These two sets of equations can be reduced to two wave equations by introducing the functions:

$$\mathbf{Z^-} = \frac{1}{r}\left(1 - \frac{2M}{r}\right)\mathbf{h_1}, \qquad \mathbf{Z^+} = \frac{r}{nr + 3M}\left[3M\mathbf{V} - r\mathbf{L}\right]$$

A wave equation for the axial and polar perturbations of a Schwarzschild black hole

$$\frac{d^2 Z^\pm}{dr_*^2} + [\omega^2 - V_\ell^\pm(r)] Z^\pm = 0$$

$$r_* = r + 2M\log(\frac{r}{2M} - 1),$$

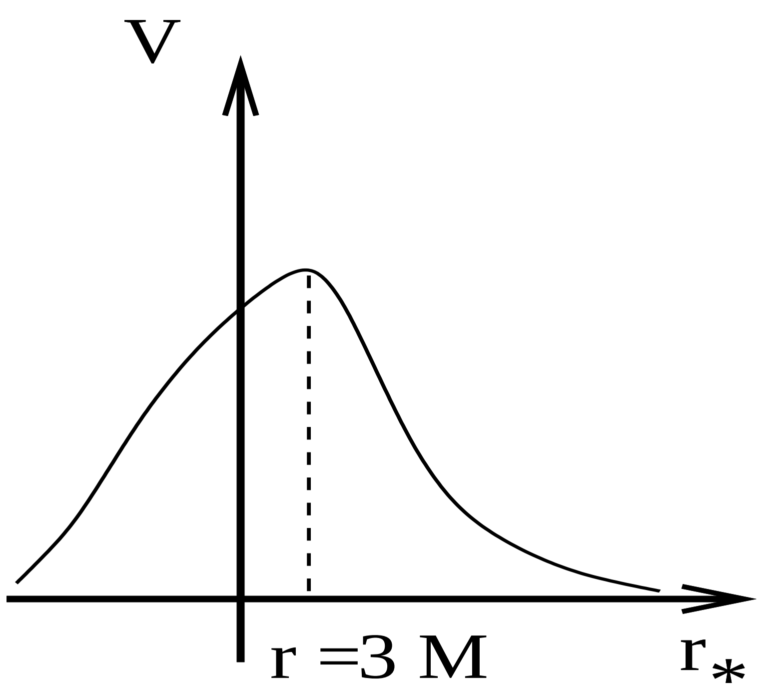
where

$$V^- \equiv V^{ax}(r) = \frac{e^{2\nu}}{r^3}[\ell(\ell+1)r + r^3 - 6M]$$

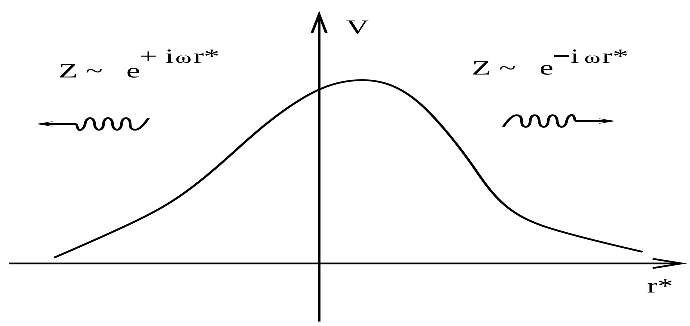
T. Regge, J. A. Wheeler, Phys. Rev. 108, 1063, 1957

$$V^+ \equiv V^{pol}(r) = \frac{2(r-2M)}{r^4(nr+3M)^2}[n^2(n+1)r^3 + 3Mn^2r^2 + 9M^2nr + 9M^3]$$

J.F. Zerilli, Phys. Rev D. **2**, *2141 (1970)*



The quasi-normal modes of a black hole are complex frequency solutions of the wave equation, which satisfy a pure outgoing wave condition at infinity, and a pure ingoing wave condition at the black hole horizon



The frequency of the lower QNM's were first found by Chandrasekhar and Detweile in 1975

$$\frac{\mathbf{d^2Z}}{\mathbf{dr_*^2}} + [\omega^2 - \mathbf{V}_\ell(\mathbf{r})]\mathbf{Z} = \mathbf{0} \qquad \text{put} \qquad \mathbf{Z} = \mathbf{e}^{\mathbf{i}\int^{\mathbf{r}*} \mathbf{\Phi dr}*}$$

$$\mathbf{i\Phi_{,r}} - \mathbf{\Phi^2} + \omega^2 - \mathbf{V} = \mathbf{0}$$

to be solved by imposing the wave condition at $r* \to \pm\infty$

$$r* \to +\infty \qquad \Phi \to -\omega, \qquad r* \to -\infty \qquad \Phi \to +\omega, \qquad \omega = \omega_0 + i\omega_i$$

	$M\omega + iM\omega$		$M\omega + iM\omega$
$\ell = 2$	0.3737+i0.0890	$\ell = 3$	0.5994+i0.0927
	0.3467+i0.2739		0.5826+i0.2813
	0.3011+i0.4783		0.5517+i0.4791
	0.2515+i0.7051		0.5120+i0.6903

For example

$$M = \mathbf{n}M_\odot, \qquad \nu_0 = \frac{12.06}{\mathbf{n}}\ kHz, \qquad \tau = \mathbf{n}5.55 \cdot 10^{-5}s$$
$$M = 10^6 M_\odot, \qquad \nu_0 = 1,21 \cdot 10^{-2}Hz, \qquad \tau = 55.5s,$$

The frequencies depend exclusively on the black hole mass

The frequencies and damping times of the axial and polar quasi normal modes are equal (they are not isospectral for stars!)

The gravitational signal emitted by a perturbed black hole will, during its last stages, decay as a superposition of these pure modes

Apart from the frequencies of the first fundamental modes which are utmost significant physically and rather easy to compute, great care must be used to determine the entire spectrum.

A **WKB approximation** has been used by
- B.F.Schutz, C.M.Will *Ap.J. Lett.* (1985)
- S.Iyer, C.M.Will *Phys. Rev.* (1987)
- S.Iyer *Phys. Rev.* (1987)
- K.D.Kokkotas, B.F.Schutz *Phys. Rev.* (1988)

to solve for the **lowest modes**
and a **higher order WKB approach** has been applied in J.W.Guinn, C.M.Will, Y.Kojima, B.F.Schutz *Class. Quantum Grav.* (1990) to find modes with **larger** n.

Method of continued fraction

Leaver E.W., *Proc. R. Soc. London A* (1985)
Leaver E.W., 1986, *J. Math. Phys.* (1986)
Leaver E.W., 1990, *Phys. Rev. D* (1990)
Leins M., Nollert H.P., Soffel M.H., *Phys.Rev. D* (1993)

For a complete bibliography:
N.Andersson *proc. R. Soc. Lond.*, **A442**, 427 (1993)

Conclusive results

- for any value of the harmonic index ℓ the real part of the frequency $\omega_{0n}(\ell)$ approaches a non zero limiting value as the order n of the mode increases.
- The imaginary part $\omega_{in}(\ell)$ increases linearly as $n/4$.
- The asymptotic behaviour is, to a high accuracy, independent of ℓ.

Perturbations of a Kerr black hole

Framework: Newman-Penrose formalism

S.Teukolsky *Phys. Rev. Lett.*, **29**, 1114 (**1972**)

S.Teukolsky *Ap. J.*, **185**, 635 (**1973**)

the equations governing the perturbations of a Kerr black hole are separated and reduced to a single master equation for the radial part of the perturbation $\mathbf{R_{lm}}$:

$$\Delta \mathbf{R_{lm,rr}} + 2(s+1)(r-M)\mathbf{R_{lm,r}} + V(r)\mathbf{R_{lm}} = 0$$
$$\Delta = r^2 - 2Mr + a^2,$$

The potential is **complex** and depends on the harmonic index **m**

$$V(r) = \frac{1}{\Delta}\Big[(r^2+a^2)^2\omega^2 - 4aMrm\omega + a^2m^2 + 2is(am(r-M) - M\omega(r^2-a^2))\Big] + \Big[2is\omega r - a^2\omega^2 - A_{lm}\Big]$$

The angular part of the perturbations, $\mathbf{S_{lm}}$, satisfies the equations of the oblate spheroidal harmonics

$$\begin{cases} [(1-u^2)\mathbf{S_{lm,u}}]_{,u} + \Big[a^2\omega^2u^2 - 2am\omega su + s + A_{lm} - \frac{(m+su)^2}{1-u^2}\Big]\mathbf{S_{lm}} = 0, \\ \mathbf{u} = \cos\boldsymbol{\theta}. \end{cases}$$

The complete perturbation is

$$\psi_s(t,r,\theta\varphi) = \frac{1}{2\pi}\int e^{-i\omega t}\sum_{l=|s|}^{\infty}\sum_{m=-l}^{l} e^{im\varphi}\mathbf{S_{lm}}(\mathbf{u})\mathbf{R_{lm}}(\mathbf{r})d\omega, \qquad (1)$$

s = **spin-weight parameter**: $s = 0, \pm1, \pm2,$ for scalar, electromagnetic and gravitational perturbations

A_{lm} is a separation constant.

Superradiance

Studying the scattering of electromagnetic waves on the potential barrier of a rotating black hole,

A.A.Starobinski, S.M.Churilov *Soviet JEPT*, **38**, 1 (1973)

W.H.Press, S.Teukolsky *Ap.J.*, **185**, 649 (1973)

found that if the incident waves have a frequency in the range

$$0 < \omega < \omega_c \qquad \text{where} \qquad \omega_c = \frac{am}{2Mr_+}, \quad m > 0, \tag{2}$$

the reflection coefficient associated to the potential barrier exceeds unity.

it is the analogue, in the domain of wave propagation, of the Penrose process in the domain of particle creation.

It occurs also when gravitational waves with a frequency in the same range are incident on the complex potential of the spin two Teukolsky equation.

Quasi-normal mode frequencies of a Kerr black hole

S.L.Detweiler *Proc. R. Soc. Lond.*, **A352**, 381 (1977)

S.L.Detweiler *Ap. J.*, **239**, 292 (1980)

E.W.Leaver *Proc. R. Soc. Lond.*, **A402**, 285 (1985)

E.Seidel, S.Iyer *Phys. Rev*, **D 41**, 374 (1990)

K.D.Kokkotas *Class.Quantum Grav.*, **8**, 2217 (1991)

the frequencies of the **qnm** depend exclusively on the **mass** and the **angular momentum** of the black hole, which are the parameters that identify the spacetime geometry.

Consequently these frequencies will be characteristic of many different processes involving the dynamical perturbations of black holes.

How do we establish if the QNM can be excited?

Test-particle approach

We assume that the perturbation is due to a test-particle with mass $\mathbf{m_0 << M}$ which follows a geodesic of the unperturbed spacetime

$$z^k(\tau) = (T(\tau), \Phi(\tau), R(\tau), \Theta(\tau)) \ ,$$

with this assumption we write Einstein's equations as

$$\delta \mathbf{G}_{\mu\nu} = \frac{\mathbf{8\pi G}}{\mathbf{c^4}} \mathbf{T}_{\mu\nu}$$

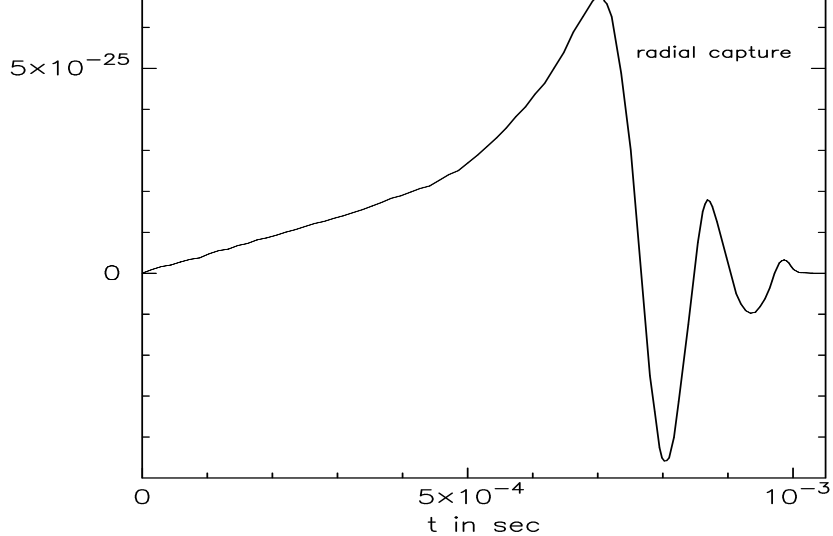
where

$$\mathbf{T}_{\mu\nu} = \mathbf{m_0} \ g_{\mu\rho} g_{\nu\sigma} \frac{dT}{d\tau} \frac{dz^\rho}{dt} \frac{dz^\sigma}{dt} \frac{\delta\left(r - R(t)\right)}{r^2} \delta^{(2)}\left(\Omega - \Omega(t)\right) \ .$$

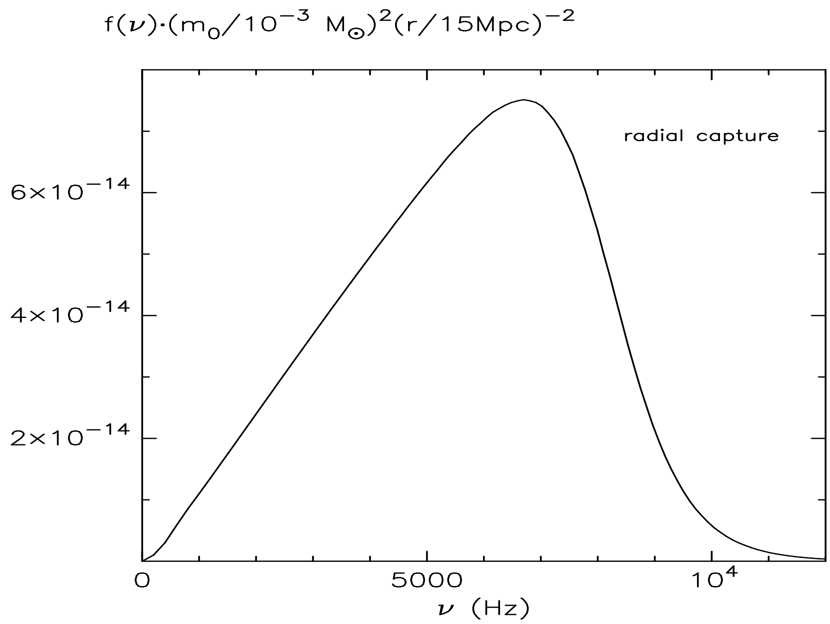
$\Omega \equiv (\vartheta, \varphi)$ is the solid angle, and
$\delta^{(2)}\left(\Omega - \Omega(t)\right) = \delta\left(\cos\vartheta - \cos\Theta(t)\right) \delta\left(\varphi - \Phi(t)\right)$

In the Regge-Wheeler-Zerilli approach, T_{ik} is expanded in tensor harmonics and the equations are separated.

M.Davis, R.Ruffini, W.H.Press, R.H.Price *Phys. Rev. Lett.*, **27**(1971)
M.Davis, R.Ruffini, J.Tiomno *Phys. Rev.*, **D5**, 2932 (1972)
V.Ferrari, R.Ruffini *Phys. Lett.* **B98**, 381 (1984)
S.L.Detweiler, E.Szedenits *Ap.J.*, **231**, 211 (1979)
K.Oohara, T.Nakamura *Phys. Lett*, **94A**, 349 (1983)
K.Oohara, T.Nakamura *Prog. Theor. Phys.*, **70**, 757 (1983)
K.Oohara, T.Nakamura *Phys. Lett*, **98A**, 407 (1983)
K.Oohara, T.Nakamura *Prog. Theor. Phys.*, **71**, 91 (1984)
T.Nakamura, M.Sasaki *Phys. Lett*, **106B**, 1627 (1981)
S.Shapiro, I.Wasserman *Ap.J.*, **260**, 838 (1982)



The $\ell = 2$ *gravitational signal emitted when a mass* m_0 *falls radially into a non rotating black hole of mass* $M = 1.5 M_\odot \gg m_0$.



The average energy flux per unit frequency, $f(\nu)$, *emitted when a mass* m_0 *falls radially into a non rotating black hole of mass* $M = 1.5 M_\odot \gg m_0$. $f(\nu)$ *is given in* $J/m^2 Hz$. M.Davis, R.Ruffini, W.H.Press, R.H.Price, *Phys. Rev. Lett.*27 n. 21 1971

$$\mathbf{f}(\nu) = < \frac{\mathbf{dE}}{\mathbf{dS d}\nu} > \equiv \frac{\mathbf{1}}{\mathbf{4}\pi \mathbf{r^2}} \int \frac{\mathbf{dE}}{\mathbf{d\Omega d}\nu} \mathbf{d\Omega}$$

The quadrupole approximation would not be able to reproduce the ringing part of the signal

Main results

• for a particle falling radially into a Schwarzschild black hole the total radiated energy is $\Delta E \sim 0.01 \left(\frac{m_0^2}{M}\right)$

• the burst ends in a ringing tail emitted when the particle coalesces into the black hole ($2 < \frac{r}{M} < 4.5$). The ringing tail can be fitted with a **linear superposition of quasi-normal modes**.

• If the particle has an initial angular momentum, the emitted energy can be increased by **up to a factor of 50**.

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If the black hole rotates

• For radial capture along the symmetry axis the energy emitted is larger than that emitted in the non rotating case. For example if $a = 0.99M$ the energy is $\simeq 1.65$ times larger than that for $a = 0$.

• if the particle falls in the equatorial plane with a vanishing angular momentum the energy is $\simeq 4.27$ times larger than that for $a = 0$.

• The particle will, in general, excite the quasi-normal modes of the black hole and more energy is emitted if the particle is **co-rotating** than if it is **counter-rotating**.

• Quasi-normal modes are not excited when a particle starting at rest at infinity is scattered by a rotating black hole

Stellar perturbations

• stars composed by a perfect fluid with assigned EOS

$$T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu - p g_{\mu\nu}$$

• adiabatic perturbations: changes in pressure and energy density arise without dissipation

• non-rotating stars

• we need to perturb Einstein's eqs. + eqs. of Hydrodynamics

$$g_{\mu\nu} = g^{\mathbf{0}}_{\mu\nu} + h_{\mu\nu}$$

and

$$\epsilon = \epsilon^{\mathbf{0}} + \delta\epsilon, \qquad p = p^{\mathbf{0}} + \delta p, \qquad u_\mu = u^{\mathbf{0}}_\mu + \delta u_\mu,$$

where $g^{\mathbf{0}}_{\mu\nu}, \epsilon^{\mathbf{0}}, p^{\mathbf{0}}, u^{\mathbf{0}}_\mu$ describe the equilibrium configuration.

The unperturbed metric $g^{\mathbf{0}}_{\mu\nu}$

$$ds^2 = e^{2\nu(r)}dt^2 - e^{2\mu_2(r)}dr^2 - r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)$$

Inside the star $\boldsymbol{\nu^0}$ and $\boldsymbol{\mu_2^0}$ have to be found by solving the equations of hydrostatic equilibrium

$$\mathbf{G}_{\mu\nu} = \frac{\mathbf{8\pi G}}{\mathbf{c^4}}\mathbf{T}_{\mu\nu},$$
$$\mathbf{T}^{\mu\nu}{}_{;\nu} = \mathbf{0}$$
$$+ \textbf{ equation of state}$$

NOTE THAT: inside the star $\boldsymbol{\nu^0}$ and $\boldsymbol{\mu_2^0}$ are related to $\boldsymbol{\epsilon^0}$ and $\mathbf{p^0}$.

Outside the star $\boldsymbol{\nu^0}$ and $\boldsymbol{\mu_2^0}$ reduce to the Schwarzschild solution.

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To find the solution for the perturbed quantities

$$h_{\mu\nu}, \quad \delta\epsilon, \quad \delta p, \quad \delta u_\mu,$$

as for the Schwarzschild black hole:

- we expand $\mathbf{h}_{\boldsymbol{\mu\nu}}$ in tensorial spherical harmonics

$\mathbf{h}_{\boldsymbol{\mu\nu}} = \mathbf{h}^{\mathbf{ax}}_{\boldsymbol{\mu\nu}\,\boldsymbol{\ell}\mathbf{m}} + \mathbf{h}^{\mathbf{pol}}_{\boldsymbol{\mu\nu}\,\boldsymbol{\ell}\mathbf{m}}$

- we choose a gauge to eliminate **4** metric components

In Addition:

- we expand $\delta\mathbf{T}_{\boldsymbol{\mu\nu}}$ in tensorial spherical harmonics

Perturbations of a non rotating star

$$\mathbf{h}^{\mathbf{ax}}_{\boldsymbol{\ell}\mathbf{m}} = \begin{pmatrix} (t) & (\varphi) & (r) & (\vartheta) \\ 0 & \mathbf{h}_0^{\mathbf{ax}} \sin\vartheta \frac{\partial Y_{\ell m}}{\partial\vartheta} & 0 & -\mathbf{h}_0^{\mathbf{ax}} \frac{1}{\sin\vartheta}\frac{\partial Y_{\ell m}}{\partial\varphi} \\ \mathbf{h}_0^{\mathbf{ax}} \sin\vartheta \frac{\partial Y_{\ell m}}{\partial\vartheta} & 0 & \mathbf{h}_1^{\mathbf{ax}} \sin\vartheta \frac{\partial Y_{\ell m}}{\partial\vartheta} & 0 \\ 0 & \mathbf{h}_1^{\mathbf{ax}} \sin\vartheta \frac{\partial Y_{\ell m}}{\partial\vartheta} & 0 & -\mathbf{h}_1^{\mathbf{ax}} \frac{1}{\sin\vartheta}\frac{\partial Y_{\ell m}}{\partial\varphi} \\ -\mathbf{h}_0^{\mathbf{ax}} \frac{1}{\sin\vartheta}\frac{\partial Y_{\ell m}}{\partial\varphi} & 0 & -\mathbf{h}_1^{\mathbf{ax}} \frac{1}{\sin\vartheta}\frac{\partial Y_{\ell m}}{\partial\varphi} & 0 \end{pmatrix}$$

$$\mathbf{h}^{\mathbf{pol}}_{\boldsymbol{\ell}\mathbf{m}} = \begin{pmatrix} (t) & (\varphi) & (r) & (\vartheta) \\ 2e^{2\nu}\mathbf{N}Y_{\ell m} & 0 & 0 & 0 \\ 0 & -2e^{2\psi}\mathbf{H_{11}} & 0 & -\mathbf{r^2V}X_{\ell m} \\ 0 & 0 & -2e^{2\mu_2}\mathbf{L}Y_{\ell m} & 0 \\ 0 & -\mathbf{r^2V}X_{\ell m} & 0 & -2e^{2\mu_3}\mathbf{H_{33}} \end{pmatrix}$$

with

$$\begin{aligned} H_{11} &= \left[\mathbf{T}Y_{\ell m} + \mathbf{V}\left(\frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} + \cot\vartheta\frac{\partial}{\partial\vartheta}\right) Y_{\ell m}\right] \\ H_{33} &= \left[\mathbf{T}Y_{\ell m} + \mathbf{V}\frac{\partial^2}{\partial\vartheta^2}\, Y_{\ell m}\right] . \end{aligned}$$

$Y_{\ell m}(\vartheta, \varphi)$ = scalar spherical harmonics

all **blue** quantities are funtions of **t and r**

$\mathbf{h}^{\mathbf{ax}}_{\boldsymbol{\ell}\mathbf{m}}$ and $\mathbf{h}^{\mathbf{pol}}_{\boldsymbol{\ell}\mathbf{m}}$ mean that when $\boldsymbol{\vartheta} \to \boldsymbol{\Pi} - \boldsymbol{\vartheta},\ \boldsymbol{\varphi} \to \boldsymbol{\Pi} + \boldsymbol{\varphi}$

$\mathbf{h^{ax}} \to \mathbf{h^{ax}}(-\mathbf{1})^{\boldsymbol{\ell}+\mathbf{1}}$ and $\mathbf{h^{pol}} \to \mathbf{h^{pol}}(-\mathbf{1})^{\boldsymbol{\ell}}$

The axial equations

$$\delta \mathbf{G}_{\vartheta\varphi} = \frac{\mathbf{8\pi G}}{\mathbf{c^4}} \delta \mathbf{T}_{\vartheta\varphi} :$$

$$\frac{1}{2}\left(\partial^2_\vartheta - \cot\vartheta\partial_\vartheta - \frac{1}{\sin\vartheta^2}\partial_{\varphi^2}\right) Y_{\ell m}(\vartheta,\varphi)\cdot$$

$$\left\{-i\omega e^{-2\nu}\mathbf{h_0} - e^{-2\mu_2}\left[\mathbf{h_{1,r}} + (\nu-\mu_2)_{,r}\mathbf{h_1}\right]\right\} = \frac{8\pi G}{c^4}\delta\mathbf{T}_{\vartheta\varphi}$$

$$\delta \mathbf{G}_{\mathbf{r}\vartheta} = \frac{\mathbf{8\pi G}}{\mathbf{c^4}} \delta \mathbf{T}_{\mathbf{r}\vartheta}$$

$$\frac{1}{2}\partial_\vartheta Y_{\ell m}(\vartheta,\varphi)\left\{e^{-2\nu}\left[\omega^2\mathbf{h_0} - i\omega\left(\mathbf{h_{0,r}} - \frac{2}{r}\mathbf{h_0}\right)\right] - \frac{2n}{r^2}\mathbf{h_1}\right.$$

$$\left. -2e^{-2\mu_2}\mathbf{h_1}\left[\nu_{,rr} + \left(\frac{1}{r}+\nu_{,r}\right)(\nu-\mu_2)_{,r}\right]\right\} = \frac{8\pi G}{c^4}\delta\mathbf{T}_{\mathbf{r}\vartheta}$$

However

$$\delta\mathbf{T}_{\vartheta\varphi} = \delta[(\epsilon+p)u_\vartheta u_\varphi - p g_{\vartheta\varphi}] = 0$$

because in the unperturbed regime only $\mathbf{u^t \neq 0}$ and

$$\frac{8\pi G}{c^4}\delta\mathbf{T}_{\mathbf{r}\vartheta} = -\frac{1}{2}\partial_\vartheta Y_{\ell m}(\vartheta,\varphi)\frac{8\pi G}{c^4}(2p\mathbf{h_1})$$

In addition from the equilibrium equations $G_{\vartheta\vartheta} = 2T_{\vartheta\vartheta}$

$$\nu_{,rr} + \left(\frac{1}{r}+\nu_{,r}\right)(\nu-\mu_2)_{,r} = +\frac{8\pi G}{c^4}pe^{2\mu_2},$$

the equations become

$$-i\omega e^{-2\nu}\mathbf{h_0} - e^{-2\mu_2}\left[\mathbf{h_{1,r}} + (\nu-\mu_2)_{,r}\mathbf{h_1}\right] = 0$$

$$e^{-2\nu}\left[-i\omega\left(\mathbf{h_{0,r}} - \frac{2}{r}\mathbf{h_0}\right) + \omega^2\mathbf{h_1}\right] - \frac{2n}{r^2}\mathbf{h_1} = 0\,.$$

The axial perturbations do not excite fluid motion. As for black holes, they are perturbations of the gravitational field!

But now $\boldsymbol{\nu}$ and $\boldsymbol{\mu_2}$ are related to $\boldsymbol{\epsilon}$ and p.

A Schroedinger Equation
For the Axial Perturbations of stars

By using the same procedure as in the case of black holes, and introducing the same function Z^{ax}

$$h_1 = re^{\mu_2-\nu}Z^{ax},$$

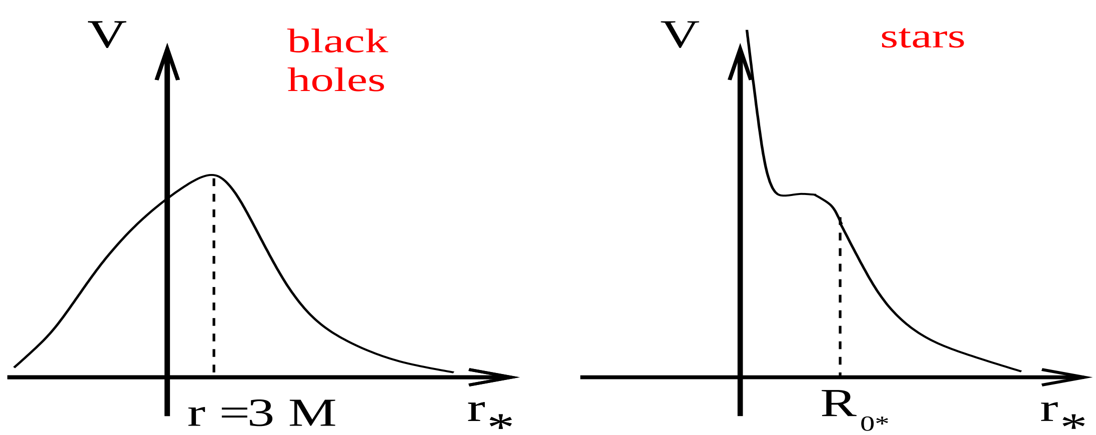
$$\begin{cases} \frac{d^2Z^{ax}}{dr_*^2} + [\omega^2 - V_\ell^{ax}(r)]Z^{ax} = 0, \\ V^{ax}(r) = \frac{e^{2\nu}}{r^3}[\ell(\ell+1)r + r^3(\epsilon - p) - 6m(r)]. \end{cases}$$

where $r_* = \int_0^r e^{-\nu+\mu_2}dr,$

S. Chandrasekhar, V. Ferrari, Proc. R. Soc. Lond. A432, 1990

The potential barrier depends on how the energy-density and the pressure are distributed inside the star in its equilibrium configuration.

The absence of fluid motion means that an incident axial wave experiences a potential scattering as it does in the case of a Schwarzschild black hole, but there is a basic difference:



The Schroedinger-like equation for a Schwarzschild black hole describes a problem of scattering by a one-dimensional potential barrier, whereas in the case of a star it describes the scattering by a central potential.

The equations for the polar perturbations couple the gravitational functions $\mathbf{N}_{\ell m}, \mathbf{T}_{\ell m}, \mathbf{V}_{\ell m}, \mathbf{L}_{\ell m}$ with the perturbations of the fluid

$$\delta \mathbf{p} \rightarrow \mathbf{\Pi}_{\ell \mathbf{m}}(\omega, \mathbf{r}) \quad \delta\epsilon \rightarrow \mathbf{E}_{\ell \mathbf{m}}(\omega, \mathbf{r})$$
$$\delta \mathbf{u_r} \rightarrow \mathbf{U}_{\ell \mathbf{m}}(\omega, \mathbf{r}) \quad \delta \mathbf{u}_\vartheta \rightarrow \mathbf{W}_{\ell \mathbf{m}}(\omega, \mathbf{r})$$

$$(\mathbf{T} - \mathbf{V} + \mathbf{N})_{,r} - \left(\frac{1}{r} - \nu_{,r}\right)\mathbf{N} - \left(\frac{1}{r} + \nu_{,r}\right)\mathbf{L} = 0,$$
$$\mathbf{V}_{,\mathbf{r},\mathbf{r}} + \left(\frac{2}{r} + \nu_{,r} - \mu_{2,r}\right)\mathbf{V}_{,\mathbf{r}} + \frac{e^{2\mu_2}}{r^2}(\mathbf{N} + \mathbf{L}) + \omega^2 e^{2\mu_2 - 2\nu}\mathbf{V} = 0,$$
$$(\mathbf{T} - \mathbf{V} + \mathbf{L}) = -\mathbf{W}$$
$$\left[\frac{d}{dr} + \left(\frac{1}{r} - \nu_{,r}\right)\right](2\mathbf{T} - k\mathbf{V}) - \frac{2}{r}\mathbf{L} = -\mathbf{U}$$
$$\frac{1}{2}e^{-2\mu_2}\left[\frac{2}{r}\mathbf{N}_{,\mathbf{r}} + \left(\frac{1}{r} + \nu_{,r}\right)(2\mathbf{T} - k\mathbf{V})_{,r} - \frac{2}{r}\left(\frac{1}{r} + 2\nu_{,r}\right)\mathbf{L}\right] +$$
$$\frac{1}{2}\left[-\frac{1}{r^2}(2n\mathbf{T} + k\mathbf{N}) + \omega^2 e^{-2\nu}(2\mathbf{T} - k\mathbf{V})\right] = \mathbf{\Pi}$$

$$\mathbf{\Pi} = -\frac{1}{2}\omega^2 e^{-2\nu}\mathbf{W} - (\epsilon + p)\mathbf{N}, \qquad \mathbf{E} = Q\mathbf{\Pi} + \frac{e^{-2\mu_2}}{2(\epsilon + p)}(\epsilon_{,r} - Qp_{,r})\mathbf{U},$$
$$\mathbf{U} = \frac{[(\omega^2 e^{-2\nu}\mathbf{W})_{,r} + (Q+1)\nu_{,r}(\omega^2 e^{-2\nu}\mathbf{W}) + 2(\epsilon_{,r} - Qp_{,r})\mathbf{N}](\epsilon + p)}{[\omega^2 e^{-2\nu}(\epsilon + p) + e^{-2\mu_2}\nu_{,r}(\epsilon_{,r} - Qp_{,r})]},$$

where

$$Q = \frac{(\epsilon + p)}{\gamma p}, \qquad \gamma = \frac{(\epsilon + p)}{p}\left(\frac{\partial p}{\partial \epsilon}\right)_{entropy=const}$$

γ is the adiabatic exponent.

Outside the star they reduce to the Zerilli equation

$$\frac{\mathbf{d^2 Z^{pol}}}{\mathbf{dr_*^2}} + [\omega^2 - \mathbf{V}_\ell^{\mathbf{pol}}(\mathbf{r})]\mathbf{Z}^{\mathbf{pol}} = \mathbf{S}^{\mathbf{pol}}(\omega, \mathbf{r})$$

$$\mathbf{V^{pol}} = \frac{\mathbf{2(r - 2M)}}{\mathbf{r^4(nr + 3M)^2}}[\mathbf{n^2(n+1)r^3 + 3Mn^2r^2 + 9M^2nr + 9M^3}]$$

Using the expression for $\mathbf{U}, \mathbf{\Pi}, \mathbf{W}$, the fluid variables can be eliminated

$$\begin{cases} \mathbf{X}_{,\mathbf{r},\mathbf{r}} + \left(\frac{2}{r} + \nu_{,r} - \mu_{2,r}\right) \mathbf{X}_{,\mathbf{r}} + \frac{n}{r^2} e^{2\mu_2} (\mathbf{N} + \mathbf{L}) + \omega^2 e^{2(\mu_2 - \nu)} \mathbf{X} = 0, \\ (r^2 \mathbf{G})_{,r} = n\nu_{,r}(\mathbf{N} - \mathbf{L}) + \frac{n}{r}(e^{2\mu_2} - 1)(\mathbf{N} + \mathbf{L}) + r(\nu_{,r} - \mu_{2,r})\mathbf{X}_{,\mathbf{r}} + \omega^2 e^{2(\mu_2-\nu)} r\mathbf{X}, \\ -\nu_{,r}\mathbf{N}_{,\mathbf{r}} = -\mathbf{G} + \nu_{,r}[\mathbf{X}_{,\mathbf{r}} + \nu_{,r}(\mathbf{N} - \mathbf{L})] + \frac{1}{r^2}(e^{2\mu_2} - 1)(\mathbf{N} - r\mathbf{X}_{,\mathbf{r}} - r^2\mathbf{G}) \\ -e^{2\mu_2}(\epsilon + p)\mathbf{N} + \frac{1}{2}\omega^2 e^{2(\mu_2-\nu)} \left\{\mathbf{N} + \mathbf{L} + \frac{r^2}{n}\mathbf{G} + \frac{1}{n}[r\mathbf{X}_{,\mathbf{r}} + (2n+1)\mathbf{X}]\right\}, \\ \mathbf{L}_{,\mathbf{r}}(1 - D) + \mathbf{L}\left[\left(\frac{2}{r} - \nu_{,r}\right) - \left(\frac{1}{r} + \nu_{,r}\right) D\right] + \mathbf{X}_{,\mathbf{r}} + \mathbf{X}\left(\frac{1}{r} - \nu_{,r}\right) + D\mathbf{N}_{,\mathbf{r}} + \\ +\mathbf{N}\left(D\nu_{,r} - \frac{D}{r} - F\right) + \left(\frac{1}{r} + E\nu_{,r}\right)\left[\mathbf{N} - \mathbf{L} + \frac{r^2}{n}\mathbf{G} + \frac{1}{n}(r\mathbf{X}_{,\mathbf{r}} + \mathbf{X})\right] = 0, \end{cases}$$

where

$$\begin{cases} A = \frac{1}{2}\omega^2 e^{-2\nu}, \qquad B = \frac{e^{-2\mu_2}\nu_{,r}}{2(\epsilon+p)}(\epsilon_{,r} - Qp_{,r}), \\ D = 1 - \frac{A}{2(A+B)} = 1 - \frac{\omega^2 e^{-2\nu}(\epsilon+p)}{\omega^2 e^{-2\nu}(\epsilon+p) + e^{-2\mu_2}\nu_{,r}(\epsilon_{,r} - Qp_{,r})}, \\ E = D(Q-1) - Q, \\ F = \frac{\epsilon_{,r} - Qp_{,r}}{2(A+B)} = \frac{2[\epsilon_{,r} - Qp_{,r}](\epsilon+p)}{2\omega^2 e^{-2\nu}(\epsilon+p) + e^{-2\mu_2}\nu_{,r}(\epsilon_{,r} - Qp_{,r})}, \end{cases}$$

These equations describe the perturbations of the gravitational field in the interior of the star, with no reference to the motion of the fluid.
To obtain the decoupled equations no assumption on the equation of state of the fluid is needed

S.Chandrasekhar, V. Ferrari Proc. R. Soc. Lond. **A432** *(1991)*

The system can be further reduced to a fourth order linear system.
J.R.Ipser, R.H.Price Phys. Rev. **D43 n.6** *(1991)*