Excerpts from Gravitational waves

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GENERATION OF GRAVITATIONAL WAVES

1.1 The quadrupole formula

A weak gravity source leads to gravitational waves governed by the wave quation

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{1.1}$$

We will now solve this equation in order to elucidate how gravitational waves are generated. For a weak gravity source, we can use the standard retarded Green's function to write the formal solution as

$$\bar{h}_{\mu\nu}(t, \vec{x}) = 4 \int \frac{T_{\mu\nu}(\vec{x}', t' = t - |\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|} d^3x \tag{1.2}$$

This solution is only valid in Lorentz gauge (since we assumed this gauge when deriving the wave equation). In general, this gauge will however not be transverse-traceless. Yet we have seen that it is useful to work in the TT gauge. So how easy is it to transform the above result into this gauge? If we are willing to sacrifice mathematical rigour, the answer is very simple: Suppose we are in a gauge where the waves propagate at the speed of light. Assume the wavefronts are nearly planar, i.e. that we are far away from the source, and orient the coordinate system in such a way that the waves propagate in the z-direction. Then

- throw away all the time-space and time-time components
- throw away all components with z indices
- keep the remaining off-diagonal terms unchanged
- remove the trace from the diagonal components

A more rigorous path to the solution would be to first project out the transverse part of $\bar{h}_{\mu\nu}$ tensor and then remove the trace. This can be done via the projection

$$q_{jk} = \delta_{jk} - n_j n_k \qquad \text{where } \vec{n} = \vec{x}/r \tag{1.3}$$

One can prove that

$$h^{\rm TT}_{jk} = q_{jl} q_{km} h_{lm} - \frac{1}{2} q_{jk} \left(q_{lm} h_{lm}\right) \tag{1.4}$$

Either way, we obtain

$$\bar{h}^{\rm TT}_{jk}(t, \vec{x}) = \left\{ 4 \int \frac{T_{jk}(\vec{x}', t' = t - |\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|} d^3x \right\}^{\rm TT} \tag{1.5}$$

and we also have $h^{\rm TT}_{\mu\nu} = \bar{h}^{\rm TT}_{\mu\nu}$.

Let us manipulate this expression somewhat. First assume that we are dealing with a slow-motion source, i.e. that $v << c \rightarrow \lambdabar >> L$. Then the source resides deep within its own "near zone" (we will discuss this concept in more detail later). Then $|\vec{x} - \vec{x}'| \approx r =$ the distance to the centre of the source, and the waves do not change much as they propagate across the source (since it is smaller than $\lambda\!\!\!^{-}$), so we have $t' = t - |\vec{x} - \vec{x}'| \approx t - r$. Thus,

$$\bar{h}^{\mathrm{TT}}_{jk}(t, \vec{x}) = \left\{ \frac{4}{r} \int T_{jk}(\vec{x}', t' = t - r) d^3x \right\}^{\mathrm{TT}} \tag{1.6}$$

To proceed we use the fact that

$$T^{\mu\nu}{}_{,\nu} = 0 \tag{1.7}$$

(where the use of partial differentiation is warranted since we are neglecting the self-gravity of the source), i.e.

$$T^{00}{}_{,0} + T^{0j}{}_{,j} = 0 \quad \text{energy conservation} \tag{1.8}$$

$$T^{j0}{}_{,0} + T^{jk}{}_{,k} = 0 \quad \text{momentum conservation} \tag{1.9}$$

After some algebra, using the fact that we can work in a global inertial frame, we have

$$T^{00}{}_{,00} x^j x^k = (T^{lm} x^j x^k)_{,lm} - 2(T^{lj} x^k + T^{lk} x^j)_{,l} + 2T^{jk} \tag{1.10}$$

This expression enables us to deduce the components of the stress energy tensor from various divergences. This is useful since, if we perform a volume integration the various divergence terms vanish, and we are left with

$$\int T^{jk} d^3x = \int T^{00}{}_{,00} x^j x^k d^3x \tag{1.11}$$

Then (1.6) reduces to

$$\bar{h}^{\mathrm{TT}jk}(t, \vec{x}) = \left\{ \frac{2}{r} \int T^{00}{}_{,00} x^{j\prime} x^{k\prime} d^3x' \right\}^{\mathrm{TT}}_{t'=t-r} = \left\{ \frac{2}{r} \frac{d^2}{dt^2} \int T^{00} x^{j\prime} x^{k\prime} d^3x' \right\}^{\mathrm{TT}}_{t'=t-r} \tag{1.12}$$

If we now define the *mass quadrupole moment* of the source as

$$M_{jk} \equiv \int \rho x_j x_k d^3x \tag{1.13}$$

and use the fact that $T^{00} = \rho$, we see that we have

$$\bar{h}^{\mathrm{TT}}_{jk} = \frac{2}{r} \ddot{M}^{\mathrm{TT}}_{jk} \tag{1.14}$$

Finally, it is customary to express this result in terms of the *reduced quadrupole moment*, which is defined by

$$I\!\!\!-_{jk} \equiv \int \rho \left(x_j x_k - \frac{1}{3} r^2 \delta_{jk} \right) d^3x \tag{1.15}$$

Simply noting that $M^{\mathrm{TT}}_{jk} = I\!\!\!-^{\mathrm{TT}}_{jk}$ (since the trace is removed by the TT operation), and reinstating G and c, our final result can be written

$$\bar{h}^{\mathrm{TT}}_{jk} = \frac{2G}{rc^4} \ddot{I\!\!\!-}^{\mathrm{TT}}_{jk}(t - r) \tag{1.16}$$

This is known that the *quadrupole formula*. As we will see later, it forms the basis for many useful gravitational-wave estimates.

1.2 Near and far-zone solutions

An analysis of gravitational wave sources often involves a number of subtle issues, and may require the use of several different approximations. The purpose of this section is to discuss some of the relevant issues by considering a particular example. Given a slow-motion source, for which $\lambdabar >> L \gtrsim M$, where L is the size of the source and M its mass, we will compute the propagation of gravitational waves from generation to exterior.

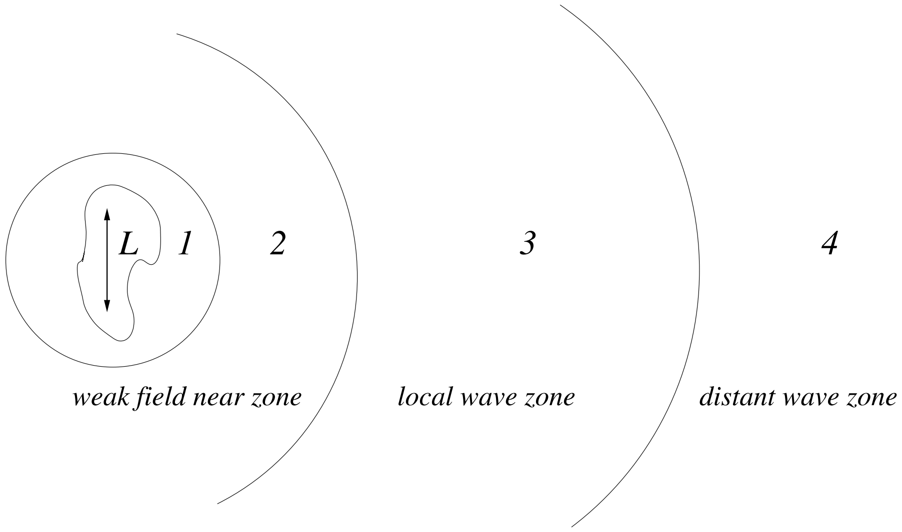

FIG. 1.1. A schematic illustration of the various regions that are relevant for the analysis of a general gravitational-wave source.

A typical situation is illustrated in Figure 1.1. We distinguish between four regions of spacetime:

1. The strong gravity region: In this region gravity is (obviously) strong. Modelling requires full nonlinear, most likely numerical, relativity.
2. The weak field near zone: Beyond a distance of $r \approx 10M$ or so we can consider gravity as weak. The problem can then be analysed using linearised theory in a flat spacetime.
3. The local wave zone: It is only in this region (and beyond) that it is meaningful to talk about gravitational "waves". The transition to the wave zone is at $r \approx \lambdabar$. Other sources of gravity are distant, and thus not important for the analysis.
4. The distant wave zone: This is the Universe at large. Geometric optics provides an adequate description of wave propagation, but other sources of gravity may be important and one must account for gravitational lensing etcetera.

Why is it necessary to consider this confusing hierarchy of spacetime regions? The answer lies in the inherent complexity of the Einstein equations. Since we cannot find exact solutions, and since the available computing power is insufficient to allow the construction of a complete numerical solution (not mentioning the technical challenges involved!), we are forced to bring every possible trick and technique to bear on each astrophysical problem of interest.

In the following we will focus on the transition between regions 2 and 3. This is a key element of the general analysis since it involves going from a region within about a wavelength of the source to the region where the concept of gravitational "waves" makes sense.

In the local wave zone we expect the gravitational-wave amplitude to i) be dominated by the mass- and current quadrupoles, and ii) fall off as the inverse distance to the source. This means that we should have

$$\bar{h}_{\mu\nu} \sim \frac{\ddot{M}_{jk} n^j n^k}{r} + \frac{\ddot{S}_{jk} n^j n^k}{r} + ... \tag{1.17}$$

where M_{jk} and S_{jk} are the mass- and current quadrupole moments, respectively.

The reason these are the leading multipoles is simple: Since mass, linear and angular momentum are all conserved, the associated multipoles (the mass monopole M, mass dipole M_j and current dipole S_j) cannot contribute.

Meanwhile, in the weak field near zone we are not dealing with radiation. Instead, the field changes quasi-statically. This follows since

$$\partial_t \sim \frac{1}{\lambdabar}\, , \partial_r \sim \frac{1}{L} \longrightarrow \partial_t << \partial_r \tag{1.18}$$

and we see that we can neglect the time-derivatives compared to the spatial ones for $\lambdabar >> L$. The upshot of this is that, in the near zone we can work with simultaneous expansions in inverse powers of r and the various multipoles. This leads to

$$\bar{h}_{\mu\nu} \sim \frac{M}{r} + \frac{M_j n^j}{r^2} + \frac{M_{jk} n^j n^k}{r^3} + ... + \frac{S_j n^j}{r^2} + \frac{S_{jk} n^j n^k}{r^3} + ... \tag{1.19}$$

In order to obtain explicit expressions we need to solve the gauge condition

$$\bar{h}^{\mu\nu}{}_{,\nu} = 0 \tag{1.20}$$

together with

$$\Box \bar{h}^{\mu\nu} = \begin{cases} 0 & \text{vacuum} \\ -16\pi T^{\mu\nu} & \text{weak internal gravity} \end{cases} \tag{1.21}$$

We have already argued that time derivatives are not important in the near zone. Keeping only the spatial derivatives we have

$$\bar{h}^{0j}{}_{,j} = 0 \tag{1.22}$$

$$\bar{h}^{jk}{}_{,k} = 0 \tag{1.23}$$

$$\nabla^2 \bar{h}^{00} = -16\pi T^{00} = -16\pi\rho \tag{1.24}$$

$$\nabla^2 \bar{h}^{0j} = -16\pi\rho v^j \tag{1.25}$$

$$\nabla^2 \bar{h}^{jk} = O(\rho v^2) \approx 0 \tag{1.26}$$

(since $\Box \to \nabla^2$ if we neglect the time derivatives). This is now a static problem, and using the standard Green's function for the Poisson equation we find that

$$\bar{h}^{00}(\vec{x}) \approx 4 \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' = -4\Phi \tag{1.27}$$

since the Newtonian gravitational potential Φ is determined from

$$\nabla^2\Phi = 4\pi\rho \tag{1.28}$$

This is turned into a multipole expansion by considering $\vec{x}'$ as a point interior to the source, while $\vec{x}$ is the exterior field point. Assuming that the centre of mass is close to $\vec{x}'$, we should have $|\vec{x}| = r >> r' = |\vec{x}'|$. Under these conditions we get

$$\begin{aligned}\frac{1}{|\vec{x}-\vec{x}'|} &= \frac{1}{\sqrt{(\vec{x}-\vec{x}')^2}} = \frac{1}{\sqrt{(r^2 - 2\vec{x}\cdot\vec{x}' + (r')^2)^2}} \\ &= \frac{1}{r}\left(1 - \frac{2\vec{n}\cdot\vec{x}'}{r} + \left(\frac{r'}{r}\right)^2\right)^{-1/2} \approx \text{ Taylor expansion} \approx \\ &\approx \frac{1}{r} + \frac{n_j x'^j}{r^2} + \frac{3}{2}\frac{n_j n_k (x'^j x'^k - (r')^2\delta^{jk}/3)}{r^3}\end{aligned} \tag{1.29}$$

From this we get

$$\bar{h}^{00}(\vec{x}) = 4\left\{\frac{1}{r}\int \rho(\vec{x}')d^3x' + \frac{n_j}{r^2}\int \rho(\vec{x}')x'^j d^3x' + \frac{3n_j n_k}{2r^3}\int \rho(\vec{x}')[x'^j x'^k - (r')^2\delta^{jk}/3]d^3x'\right\} \tag{1.30}$$

Here we can identify the first integral as the mass (monopole moment) M, the second integral is the mass dipole moment M_j and the third integral is the quadrupole moment M_{jk} (with the trace removed). In other words, we have shown that

$$\bar{h}^{00} = 4\left\{\frac{M}{r} + \frac{M_j n^j}{r^2} + \frac{3}{2}\frac{\mathcal{I}_{jk} n^j n^k}{r^3}\right\} \tag{1.31}$$

in the near zone.

As we extend this result into the wave zone, the static parts (the monopole and the dipole) retain their form. But the quadrupole part (which is dynamic) no longer takes the form in (1.31). As r increases we must account for retardation effects and describe the quadrupole component in terms of outgoing gravitational waves.

We thus want to match (1.31) to an expression that is valid in the wave zone. For obvious reasons we focus on the quadrupole term. We need to determine $\bar{h}^{\mu\nu}$ in regions 2 and 3 of Figure 1.1. In those regions we cannot neglect the time derivatives, but on the other hand gravity is weak so we can linearise the equations. Thus we need to find a solution to

$$\bar{h}^{\mu\nu}{}_{,\nu} = 0 \tag{1.32}$$

$$\Box\bar{h}^{\mu\nu} = 0 \tag{1.33}$$

which satisfies the boundary condition

$$\bar{h}^{00} = \frac{6\mathcal{I}_{jk} n^j n^k}{r^3} \qquad \text{for } r << \lambda\!\!\!^{-} \tag{1.34}$$

and which corresponds to purely outgoing waves as $r \to \infty$.

From the analysis in the previous section, we already know what form we expect the solution to take. This provides a short cut to the required answer. Instead of writing down the general solution we note that the scalar wave equation $\Box\psi = 0$ admits the outgoing wave solution $\psi = f(t-r)/r$ where f can be any function. In our problem (and in a local inertial frame) each component of $\bar{h}_{\mu\nu}$ should satisfy the scalar wave equation. This inspires us to write the solution as

$$\bar{h}^{00} = 2\left[\frac{1}{r}\mathcal{I}_{jk}(t-r)\right]_{,jk} \quad (1.35)$$

Expanding this expression for small r, using

$$\left(\frac{1}{r}\right)_{,j} = -\frac{n_j}{r^2} = -\frac{x_j}{r^3} \quad (1.36)$$

$$\left(\frac{1}{r}\right)_{,jk} = -\frac{3}{r^3}\left[n_j n_k - \frac{\delta_{jk}}{3}\right] \quad (1.37)$$

and

$$[\mathcal{I}_{jk}(t-r)]_{,j} \sim \frac{\mathcal{I}_{jk}}{\lambda\!\!\!^{-}} \text{ where } \lambda\!\!\!^{-} >> r \quad (1.38)$$

we see that this solution satisfies the required boundary condition (1.34) in the near zone.

Given an expression for $\bar{h}^{00}$ we can readily determine all other components of the perturbed metric as follows. First we integrate

$$\bar{h}^{00}{}_{,0} = -\bar{h}^{0j}{}_{,j} \quad (1.39)$$

to get

$$\bar{h}^{0j} = -2\left[\frac{1}{r}\dot{\mathcal{I}}^{jk}(t-r)\right]_{,k} \quad (1.40)$$

and then use

$$\bar{h}^{jk}{}_{,k} = -\bar{h}^{j0}{}_{,0} = -\bar{h}^{0j}{}_{,0} \quad (1.41)$$

to find that

$$\bar{h}_{jk} = \frac{2}{r}\ddot{\mathcal{I}}_{jk}(t-r) \quad (1.42)$$

This is the main result. Retaining only the TT part in the wave zone we have

$$\bar{h}^{\rm TT}_{jk} = \frac{2}{r}\ddot{\mathcal{I}}^{\rm TT}_{jk} \quad (1.43)$$

Not surprisingly, we have agreed at the same expression for the gravitational-wave field as in the previous section. Yet there are some subtle differences. In particular, the approach we used in this section remains valid also when we are dealing with a source with strong internal gravity. For example, we can still deduce the gravitational wave strength from the quadrupole moment in the weak field regime. The difference is that in that case we must use the full nonlinear theory to model the interior dynamics.

In the above calculation we focussed on the mass quadrupole. The method can (at the cost of increased algebraic complexity) be extended to the full multipole expansion. Here we will simply comment on the results for the leading current multipole. The current quadrupole moment radiation follows from

$$\bar{h}^{0j} = -4\frac{\hat{n}\times\vec{S}}{r^2} \tag{1.44}$$

where

$$\vec{S} = \int \vec{x}\times\rho(\vec{x})\vec{v}(\vec{x})d^3x \tag{1.45}$$

is the angular momentum of the source. Alternatively, this can be written

$$\bar{h}^{0j} = \epsilon_{jab}n^a n^c S^b{}_c \tag{1.46}$$

where we have defined

$$S_{bc} = \left[\int (\vec{x}\times\rho\vec{v})^b x^c d^3x\right]^{\rm STF} \tag{1.47}$$

and where STF means that we have symmetrised on b and c and removed the trace, i.e. the object is Symmetric and Trace-Free. Carrying out the required algebra we find that

$$h^{\rm TT}_{jk} = -\frac{8}{3}\left[\frac{1}{r}\epsilon_{jab}n^a\ddot{S}^b{}_k(t-r)\right]^{\rm STF} \tag{1.48}$$

Having obtained the key multipole formulas, it is useful to estimate the expected relative magnitudes. Consider a source of size L, with mass M and characteristic internal velocity v. Assume that the gravitational waves have wavelength $\lambdabar$ and that the distance to the observer is r. Then we see that the mass quadrupole radiation would have magnitude

$$h \sim \frac{ML^2}{r\lambdabar^2} \sim \frac{Mv^2}{r} \tag{1.49}$$

i.e. it would be of the order of the internal kinetic energy (associated with the non-axisymmetric motion) of the source. Meanwhile we find for the current quadrupole

$$h \sim \frac{ML^2 v}{r\lambda^2} \sim \left(\frac{Mv^2}{r}\right)v \tag{1.50}$$

If we recall that we used geometric units ($c = 1$) we see that the current quadrupole radiation will be smaller than the mass quadrupole by a factor of v/c. Thus we conclude that the main contribution to the gravitational radiation emission comes from variations in the mass quadrupole. There are, however, sources where this conclusion does not hold. A typical example is the r-mode oscillations of rotating neutron star. In that case, the sloshing of the waves is such that the mass multipoles are small and the radiation is instead dominated by the current multipoles. This happens because the motion is such that the density of the star is unchanged to leading order.

To conclude this section, we can work out expressions for the radiated energy and angular momentum. The gravitational-wave luminosity follows by integrating the flux in the local wave zone. This leads to (with the factors of G and c restored)

$$\frac{dE}{dt} = -\frac{G}{5c^5}\left\langle \dddot{\mathcal{I}}_{jk}\dddot{\mathcal{I}}_{jk}\right\rangle \tag{1.51}$$

and

$$\frac{dS_j}{dt} = -\frac{2}{5}\epsilon_{jab}\left\langle \ddot{\mathcal{I}}_{ac}\dddot{\mathcal{I}}_{bc}\right\rangle \tag{1.52}$$

1.3 Binary inspiral

The archetypal gravitational-wave source is a compact binary system. Gravitational waves are emitted as the stars (or black holes) orbit each other and as a result the binary separation decreases. In contrast to in Newtonian gravity the two-body problem remains unsolved in General Relativity. Given the lack of suitable exact solutions to the Einstein field equations significant effort has gone into developing various approximations and numerical approaches to the problem. We can readily use the formulas derived in the previous sections to estimate the radiation from this kind of system.

Consider a binary system with individual masses M_1 and M_2 and separation $2R$. Introduce the total and reduced masses

$$M = M_1 + M_2 \qquad \text{and} \qquad \mu = \frac{M_1 M_2}{M} \tag{1.53}$$

such that

$$M_1 a_1 = M_2 a_2 = \mu R \tag{1.54}$$

Assume that the system is oriented as in Figure 1.3, i.e. that the stars rotate around the z-axis of a Cartesian coordinate system whose origin coincides with the systems centre of mass. Let the angle to the x-axis be $\phi = \Omega t$, where Ω is the rotation frequency.

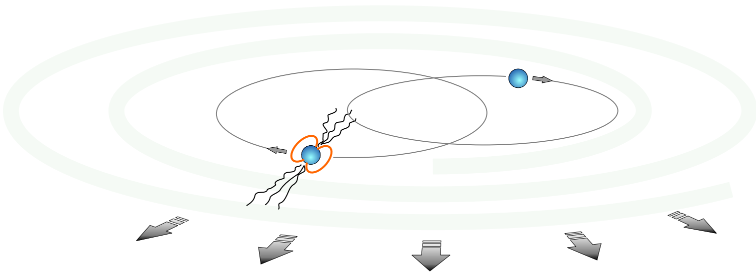

FIG. 1.2. A schematic illustration of the binary pulsar PSR 1913+16. The orbit shrinks as it loses energy due to gravitational-wave emission.

Working out the required components of the quadrupole moment, we have

$$\mathcal{I}_{xx} = (M_1 a_1^2 + M_2 a_2^2)\left[\cos^2\phi - \frac{1}{3}\right] = \frac{\mu R^2}{2}\cos 2\phi + \text{ constant} \tag{1.55}$$

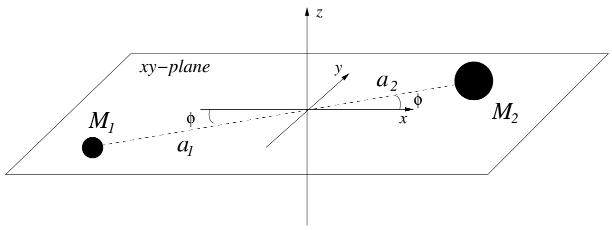

FIG. 1.3. Illustrating the calculation in the text.

We do not need to worry about the constant piece since we will take several time-derivatives in order to arrive at the gravitational-wave field. The time-varying contributions are

$$\mathcal{I}_{xx} = -\mathcal{I}_{yy} = \frac{\mu R^2}{2}\cos 2\phi \tag{1.56}$$

$$\mathcal{I}_{xy} = \mathcal{I}_{yx} = \frac{\mu R^2}{2}\sin 2\phi \tag{1.57}$$

Then we find that

$$\begin{aligned} \frac{dE}{dt} &= \frac{G}{5c^5}\left\langle \dddot{\mathcal{I}}_{jk}\dddot{\mathcal{I}}^{jk}\right\rangle = \\ &= \frac{G}{5c^5}\left(\frac{\mu R^2}{2}\right)^2 (2\Omega)^6 \left\langle 2\sin^2 2\phi + 2\cos^2 2\phi\right\rangle = \\ &= \frac{32}{5}\frac{G}{c^5}\mu^2 R^4 \Omega^6 \end{aligned} \tag{1.58}$$

Finally, using the fact that the orbital rotation frequency can be determined from Kepler's law, i.e.

$$\Omega^2 = \frac{M}{R^3} \tag{1.59}$$

and introducing the so-called "chirp mass"

$$\mathcal{M} = \mu^{3/5} M^{2/5} \tag{1.60}$$

we have

$$\frac{dE}{dt} = \frac{32}{5}(\mathcal{M}\Omega)^{10/3} \tag{1.61}$$

The above formula tells us at what rate gravitational waves carry energy away from the system. In order to predict the rate at which the binary orbit shrinks as a result we need to estimate to total energy of the system. The energy readily follows as

$$E = \frac{M_1 v_1^2}{2} + \frac{M_2 v_2^2}{2} - \frac{M_1 M_2}{R} = -\frac{\mu M}{2R} = -\frac{\mathcal{M}^{5/3}\Omega^{2/3}}{2} \tag{1.62}$$

From this we find that the period of the system P changes as

$$\frac{\dot{P}}{P} = -\frac{\dot{\Omega}}{\Omega} = \frac{3}{2}\frac{\dot{R}}{R} = -\frac{3}{2}\frac{\dot{E}}{E} \tag{1.63}$$

Accounting for the fact that (1.61) approximates the energy carried away by gravitational waves, while the change in the orbit follows by removing the corresponding energy from the system, we find that

$$\frac{\dot{P}}{P} = -\frac{96}{5}\mathcal{M}^{5/3}\Omega^{8/3} \tag{1.64}$$

One of the best tests of General Relativity, and also the strongest indirect evidence of the existence of gravitational radiation is provided by the *binary pulsar* 1913+16. It is a double neutron star system with an orbital period of about 7 hours and 45 mins. We can use the above formulas to estimate the rate at which the orbital period ought to change because of the fact that gravitational waves are emitted. If we assume that both companions have a mass of $1.4M_\odot$ (which turn out to be almost right) and that the orbit is circular we would predict that

$$\dot{P} \approx -2 \times 10^{-13} \text{ s/s} \tag{1.65}$$

This should be compared to the observed value

$$\dot{P} \approx -2.3 \times 10^{-12} \text{ s/s} \tag{1.66}$$

At first sight our prediction does not seem very impressive. But in reality the orbit is far from circular as we have assumed. For an elliptic orbit one can show that our formulas will pick up a multiplicative factor which depends on the ellipticity ϵ. This leads to

$$\dot{E} = f(\epsilon)\dot{E}_{\text{circular}} \tag{1.67}$$

with

$$f(\epsilon) = \left[1 + \frac{73}{24}\epsilon^2 + \frac{37}{96}\epsilon^4\right](1-\epsilon^2)^{-7/2} \tag{1.68}$$

In the case of the binary pulsar, $\epsilon = 0.617$ which means that $f(\epsilon) \approx 11.8$. Hence, accounting for the ellipticity brings the theoretical estimate much closer to the observed result. In fact, after more than 25 years of timing of the binary pulsar the observations now agree with the predictions of General Relativity to within 1%.

For an elliptical orbit, the eccentricity evolves. Gravitational wave are emitted primarily at periastron. Hence, this is where gravitational radiation reaction has the greatest effect on the orbit. One can prove that the end result is that the orbit becomes more circular as the orbit shrinks.

As an alternative, we can write down a differential equation for the rate of change of the binary separation. Solving this equation we find that

$$R = R_0\left(1 - \frac{t}{t_m}\right)^{1/4} \tag{1.69}$$

where

$$t_m = \frac{5}{256}\frac{R_0^4}{\mu M^2} = \frac{5}{256}\frac{M^2}{\mu(M\Omega)^{8/3}} \tag{1.70}$$

Using the fact that the gravitational-wave frequency is twice the rotation rate,

$$f = \frac{\Omega}{\pi} \tag{1.71}$$

our formula suggests that

$$t_m \approx 3\text{s}\left(\frac{f}{100\text{Hz}}\right)^{-8/3}\left(\frac{10M_\odot}{\mathcal{M}}\right)^{5/3} \tag{1.72}$$

This provides a useful estimate of the time that remains until merger (even though the merger will not correspond to exactly $R = 0$). A sample of estimated t_m for the types of binaries that are expected to be the most relevant for ground based detectors are given in Table 1.3. From this data we see that all binaries that are likely to be observed by (say) LIGO will merge well within a year from the time when they first enter the detector bandwidth.

f	$1.4 - 1.4\ M_\odot$	$1.4 - 10\ M_\odot$	$10 - 10\ M_\odot$
1 Hz	5 days	1 day	4.8 hrs
10 Hz	17 min	4 min	38 s
100 Hz	2 s	0.5 s	0.08 s

Table 1.1 *Estimated times until merger (t_m) for various classes of binary systems, the most common of which are expected to be double neutron star (mass about $1.4M_\odot$) systems, double $10M_\odot$ black hole binaries, and a mixture of the two. f is the gravitational-wave frequency of the signal when the system is first observed.*

Finally, let us estimate the actual gravitational-wave amplitude. This is a little bit more complicated than the above analysis because it requires us to specify a given observer. The main reason for this is the fact that any given observer can only see waves due to transverse motion according to his/her location visavi the source. Let us assume that the observer is situated a distance r away from the source on the y-axis. Suppose we want to work out h_{xx}^{TT}. Then we use $n = \hat{e}_y$ and the TT-projection to get

$$\mathcal{I}_{xx}^{\text{TT}} = q_{xl}q_{xm}\mathcal{I}_{lm} - \frac{1}{2}q_{xx}\left(q_{lm}\mathcal{I}_{lm}\right) = \frac{1}{2}\mathcal{I}_{xx} \tag{1.73}$$

This then leads to

$$h_{xx}^{\text{TT}} = -\frac{2\mu R^2\Omega^2}{r}\cos 2\Omega(t-r) = -\frac{2\mathcal{M}^{5/3}\Omega^{2/3}}{r}\cos 2\Omega(t-r) \tag{1.74}$$

From our various results we see that the chirp-mass $\mathcal{M}$ plays a key role. In fact, we find that this is the only combination of the two masses that can be infered from an observed gravitational-wave signal (at this level of approximation). However, we also see that if one can observe both the shrinkage of the orbit and the gravitational-wave field (h_{xx}^{TT}, say) then one can infer both the chirp mass and the distance to the source. This means that coalescing binaries are "standard candles" which may be used to infer eg. the Hubble constant. By extracting higher order post-Newtonian terms one can also hope to infer the individual masses, the spins and maybe put constraints on the graviton mass.

1.4 Gravitational-wave back reaction

Up to this point, we have discussed the gravitational wave energy loss in terms of the radiation that reaches a distant observer. It is important to note that we never required any information about the direct effect that the emission of the waves might have on the source. We are, in fact, fortunate to be able to discuss the problem in this somewhat indirect way, because accounting for the gravitational-wave back reaction is far from easy. That this is the case is more or less obvious since (i) gravitational waves typically have wavelengths larger than the size of the radiating system, and (ii) we can only localize the energy carried by the waves to one wavelength or so.

Despite these conceptual difficulties it is often desirable to model the direct back reaction that the waves have on the source. This would certainly be the case if we wanted to descibe a system that changes considerable as the radiation is emitted, i.e. when averaging over one wavelength may be somewhat dubious. Typical astrophysical systems where we need this kind of modelling would be the eccentric orbit inspiral of a small body into a rapidly rotating Kerr black hole, and the growth of unstable oscillations of a rotating neutron star.

It is common to model the radiation reaction acting on a body of mass m as a local force

$$F_j = -m\nabla_j \phi^{\mathrm{R}} \tag{1.75}$$

where

$$\phi^{\mathrm{R}} = \frac{G}{5c^5}\mathcal{I}^{(5)\mathrm{TT}}_{jk} x^j x^k \tag{1.76}$$

where (5) indicates the fifth time derivative.

In order to be consistent (which is the least that we should require) this formula must lead to the energy loss we predict from the quadrupole formula. For the particular case of a fluid source, we can show this in the following way: First note that the acceleration that the radiation reaction force would impart in a Newtonian source is

$$a_j = -\nabla_j \phi^{\mathrm{R}} \tag{1.77}$$

Then the associated kinetic energy change follows from

$$E = \int \frac{\rho v^2}{2} d^3x \longrightarrow \dot{E} = \int \rho v^j a_j d^3x \tag{1.78}$$

Next we note that

$$\mathcal{I}^{\mathrm{TT}}_{jk} = I^{\mathrm{TT}}_{jk} = \left[\int \rho x_j x_k d^3x\right]^{\mathrm{TT}} \tag{1.79}$$

This means that, assuming that we work in TT-gauge (and omitting the TT superscript from now on), we have

$$\dot{E} = -\int \rho v^j \left[\frac{1}{5}\nabla_j \left\{\left(\frac{\partial^5 \mathcal{I}_{lm}}{\partial t^5}\right) x^l x^m\right\}\right] d^3x = \text{ integrate by parts } =$$
$$= \frac{1}{5}\int \left(\frac{\partial^5 \mathcal{I}_{lm}}{\partial t^5}\right) x^l x^m \nabla_j(\rho v^j) d^3x = \text{ use the continuity equation} =$$
$$= \frac{1}{5}\int \frac{\partial \rho}{\partial t}\left(\frac{\partial^5 \mathcal{I}_{lm}}{\partial t^5}\right) x^l x^m d^3x =$$
$$= \frac{1}{5}\int\int \frac{\partial \rho(x)}{\partial t} x^l x^m \frac{\partial^5}{\partial t^5}\left[\rho(x')x'_l x'_m\right] d^3x' d^3x \tag{1.80}$$

Integrating over the entire body we have

$$\dot{E} = \frac{1}{5}\frac{\partial \mathcal{I}^{jk}}{\partial t}\frac{\partial^5 \mathcal{I}_{jk}}{\partial t^5} \tag{1.81}$$

At this point we recall that the rate of energy loss we deduced from the quadrupole formula required averaging over at least one orbit. Carrying out this averaging essentially corresponds to integrating in time over an entire period, and we can readily use integration by parts to show that

$$\left\langle \dot{E} \right\rangle = \frac{1}{5}\left\langle \frac{\partial \mathcal{I}^{jk}}{\partial t}\frac{\partial^5 \mathcal{I}_{jk}}{\partial t^5}\right\rangle = -\frac{1}{5}\left\langle \frac{\partial^2 \mathcal{I}^{jk}}{\partial t^2}\frac{\partial^4 \mathcal{I}_{jk}}{\partial t^4}\right\rangle = \frac{1}{5}\left\langle \dddot{\mathcal{I}}^{jk}\dddot{\mathcal{I}}_{jk}\right\rangle \tag{1.82}$$

Finally, if we account for the simple fact that this formula represents the energy change in the system, while our previous result described the energy carried away by the waves we see that the two pictures are, indeed, consistent.

2

THE POST-NEWTONIAN APPROXIMATION

So far we have only discussed inherently linear methods we have discussed for modelling gravitational-wave sources. Yet general relativity is a fundamentally nonlinear theory. It is natural to worry that our various approaches are somewhat flawed, and that we may be missing out on some key physics. In this chapter we will discuss an interative approach to the full non-linear problem, the so-called post-Newtonian approximation.

2.1 The relaxed Einstein equations

We begin by recasting the full Einstein equations

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = 8\pi T^{\alpha\beta} \tag{2.1}$$

into what is usually refered to as their "relaxed" form. To do this we define the potential

$$h^{\alpha\beta} \equiv \eta^{\alpha\beta} - (-g)^{1/2}g^{\alpha\beta} \tag{2.2}$$

and choose a coordinate system defined by the Lorentz gauge condition

$$h^{\alpha\beta}{}_{,\beta} = 0 \tag{2.3}$$

With these definitions the Einstein equations (2.1) take the form [EXPAND ON THIS!]

$$\Box_\eta h^{\alpha\beta} = -16\pi\tau^{\alpha\beta} \ , \tag{2.4}$$

where $\Box_\eta \equiv -\partial^2/\partial t^2 + \nabla^2$ is the flat-spacetime wave operator. The source on the right-hand side is given by the effective stress-energy pseudotensor [EXPLAIN PSEUDOTENSOR]

$$\tau^{\alpha\beta} = (-g)T^{\alpha\beta} + (16\pi)^{-1}\Lambda^{\alpha\beta} \ , \tag{2.5}$$

where $\Lambda^{\alpha\beta}$ is the non-linear contribution of the field itself. It is explicitly given by

$$\Lambda^{\alpha\beta} = 16\pi(-g)t^{\alpha\beta}_{\rm LL} + (h^{\alpha\mu}{}_{,\nu}h^{\beta\nu}{}_{,\mu} - h^{\alpha\beta}{}_{,\mu\nu}h^{\mu\nu}) \tag{2.6}$$

and $t^{\alpha\beta}_{\rm LL}$ is the so-called *Landau-Lifshitz pseudotensor*, defined by

$$\begin{aligned} 16\pi(-g)t^{\alpha\beta}_{\rm LL} \equiv\ & g_{\lambda\mu}g^{\nu\rho}h^{\alpha\lambda}{}_{,\nu}h^{\beta\mu}{}_{,\rho} + \frac{1}{2}g_{\lambda\mu}g^{\alpha\beta}h^{\lambda\nu}{}_{,\rho}h^{\rho\mu}{}_{,\nu} - 2g_{\mu\nu}g^{\lambda(\alpha}h^{\beta)\nu}{}_{,\rho}h^{\rho\mu}{}_{,\lambda} \\ & + \frac{1}{8}(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu})(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau})h^{\nu\tau}{}_{,\lambda}h^{\rho\sigma}{}_{,\mu} \ . \end{aligned} \tag{2.7}$$

One can show that, by virtue of the gauge condition (2.3), the source term satisfies the conservation law

$$\tau^{\alpha\beta}{}_{,\beta} = 0 \tag{2.8}$$

which is equivalent to the equation of motion of the matter

$$T^{\alpha\beta}{}_{;\beta} = 0. \tag{2.9}$$

It is important to realize that Equation 2.4) is exact. We have made no approximations in the derivation. The equation is equivalent to the Einstein equations provided that spacetime can be covered by harmonic coordinates. This alternative form of the relativistic equations is called the "relaxed" Einstein equations because one can write down a formal solution as a functional of source variables without specifying the motion of the source. After doing this, the harmonic gauge condition, Eq. (2.3) or the equations of motion, Eq. (2.9) are imposed to determine the metric as a function of spacetime.

Of course, it is not easy to find an actual solution to these equations. Especially since the "source" in Eq. (2.4) contains terms which depend explicitly on $h^{\alpha\beta}$. This is of course the quantity we are trying to solve for.

A further complication arises from the fact that $\tau^{\alpha\beta}$ depends on the fields $h^{\alpha\beta}$ which are likely to have infinite spatial extent. After all, these terms represent the outgoing gravitational waves that we are interested in. As is clear from (2.4), these waves will serve as a contribution to the source, thus generating an additional component of the radiation.

Finally, it should be noted that the second derivative term $h^{\alpha\beta}{}_{,\mu\nu}h^{\mu\nu}$ really belongs on the left-hand side with the other second derivative terms in the wave operator (it contributes to the principal part of the differential operator). This term modifies the propagation characteristics of the field from the flat-spacetime characteristics represented by the d'Alembertian operator to those of the true null cones of the curved spacetime around the source, which will deviate from the flat null cones of the harmonic coordinates.

2.2 Direct integration approach

The formal solution to Eq. (2.4) with an outgoing-wave boundary condition can be written down in terms of the retarded flat-space Green function:

$$h^{\alpha\beta}(t,\mathbf{x}) = 4\int \frac{\tau^{\alpha\beta}(t',\mathbf{x}')\delta(t'-t+|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} d^4x' \ , \tag{2.10}$$

However, given that the function we are in fact solving for forms a part of the integrand on the right-hand side, we have really only converted the differential equation into an integral equation. The equation now represents an integration of $\tau^{\alpha\beta}/|\mathbf{x}-\mathbf{x}'|$ over the past harmonic null cone $\mathcal{C}$ emanating from the field point $(t,\mathbf{x})$, see Fig. 2.1. T his past null cone intersects the world tube $\mathcal{D}$ enclosing the near zone at the three-dimensional hypersurface $\mathcal{N}$. This means that the integral of Eq. (2.10) consists of two pieces, an integration over the hypersurface $\mathcal{N}$, and an integration over the rest of the past null cone $\mathcal{C}-\mathcal{N}$.

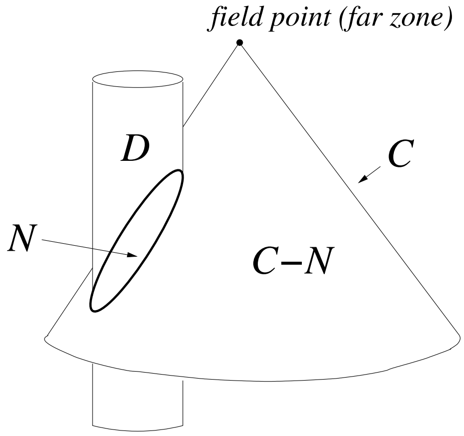

Fig. 2.1. The past harmonic null cone $\mathcal{C}$ of an exterior field point intersects the near zone world tube $\mathcal{D}$ at a hypersurface $\mathcal{N}$.

Let us focus on the case when the field point is in the far zone, eg. the case that is relevant if we want to calculate the gravitational-wave signal. Then we first carry out the t' integration in Eq. (2.10), to obtain

$$h^{\alpha\beta}_{\mathcal{N}}(t,\mathbf{x}) = 4\int_{\mathcal{N}} \frac{\tau^{\alpha\beta}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \ . \tag{2.11}$$

Within the near zone, the spatial integration variable $\mathbf{x}'$ satisfies $|\mathbf{x}'| \leq \mathcal{R} < r$, where the distance to the field point $r = |\mathbf{x}|$. Expanding the x'-dependence in both occurrences of $|\mathbf{x}-\mathbf{x}'|$ in the integrand in powers of $|\mathbf{x}'|/r$, it is straightforward to show that

$$h^{\alpha\beta}_{\mathcal{N}}(t,\mathbf{x}) = 4\sum_{q=0}^{\infty} \frac{(-1)^q}{q!} \partial_Q \left(\frac{1}{r} M^{\alpha\beta Q}(u) \right) \ , \tag{2.12}$$

where

$$M^{\alpha\beta Q}(u) \equiv \int_{\mathcal{M}} \tau^{\alpha\beta}(u,\mathbf{x}') x'^Q d^3x' \ . \tag{2.13}$$

In Eqs. (2.12) and (2.13), the index Q is a multi-index, such that $\partial_Q \equiv \partial_{i_1}\partial_{i_2}\ldots\partial_{i_q}$ and the superscript Q in $M^{\alpha\beta Q}$ denotes $i_1 i_2 \ldots i_q$, with summation over repeated indices assumed. The integrations in Eq. (2.13) are now over the hypersurface $\mathcal{M}$, which is the intersection of the near-zone world-tube with the constant-time hypersurface $t_{\mathcal{M}} = u = t - r$. Roughly speaking, each term in the Taylor series is smaller than its predecessor by a factor of order $v \ll 1$, provided we restrict attention to slow-motion sources.

In the case of the outer integral, we make a change of integration variable from (r', θ', ϕ') to (u', θ', ϕ'), where

$$t - u' = r' + |\mathbf{x}-\mathbf{x}'| \, , \tag{2.14}$$

we can write the integral over the rest of the past null cone $\mathcal{C} - \mathcal{N}$ in the form

$$h^{\alpha\beta}_{\mathcal{C}-\mathcal{N}}(t, \mathbf{x}) = 4 \int_{-\infty}^{u} du' \oint_{\mathcal{C}-\mathcal{N}} \frac{\tau^{\alpha\beta}(u' + r', \mathbf{x}')}{t - u' - \hat{\mathbf{n}}' \cdot \mathbf{x}} [r'(u', \Omega')]^2 d^2\Omega' \ , \tag{2.15}$$

where, from Eq. (2.14)

$$r'(u', \Omega') = [(t - u')^2 - r^2]/[2(t - u' - \hat{\mathbf{n}}' \cdot \mathbf{x})] \ . \tag{2.16}$$

This change of variables represents an integration first over the two-dimensional intersection of the past null cone $\mathcal{C}$ with the future null cone $t' = u' + r'$ emanating from the center of mass of the system at $t_{\rm CM} = u'$ (Fig. **??**), followed by the u'-integration over all such future-directed cones, starting from the infinite past, and terminating in the cone emanating from the center of mass at time u, which is tangent to the past null cone of the observation point.

The general strategy is now as follows: Because the field $h^{\alpha\beta}$ appears in the source of the field equation, the usual method of solution is to iterate: substitute $h^{\alpha\beta} = 0$ in the right-hand side of Eq. (2.10) and solve for the first-iterated ${}_1h^{\alpha\beta}$; substitute that into Eq. (2.10) and solve for the second-iterated ${}_2h^{\alpha\beta}$, and so on (imposing the gauge condition Eq. (2.3) consistently at each order). The general sequence of iterations is shown shematically in Fig. **??**. The matter variables m_A and the $(N-1)$-iterated field ${}_{N-1}h^{\alpha\beta}$ are used to determine ${}_{N-1}T^{\alpha\beta}$ and ${}_{N-1}\Lambda^{\alpha\beta}$. Eq. (2.10) then yields ${}_Nh^{\alpha\beta}$ as a function of spacetime and a functional of the matter variables. Then, if one wishes to determine the motion of the source, one substitutes ${}_Nh^{\alpha\beta}$ into the matter stress-energy tensor, and obtains the equations of motion from ${}_N\nabla_\beta({}_NT^{\alpha\beta}) = 0$ where ${}_N\nabla_\beta$ denotes the covariant derivative using the Nth iterated field. If one wishes to determine the Nth iterated gravitational field as a function of spacetime (i.e. with the matter variables determined as functions of spacetime to a consistent order), then one only needs to solve the equations of motion ${}_{N-1}\nabla_\beta({}_{N-1}T^{\alpha\beta}) = 0$, which are equivalent to the N-th iterated gauge condition ${}_Nh^{\alpha\beta}{}_{,\beta} = 0$.

At the first iteration, the solution is simply linearized general relativity. With ${}_0h^{\alpha\beta} = 0$ substituted into the right-hand-side of Eq. (2.10), the outer integrals vanish, and the inner integrals over the special relativistic $T^{\alpha\beta}$ have compact support. There is no $\mathcal{R}$-dependence in the integrals, trivially. For field points outside the source ($|\mathbf{x}| > |\mathbf{x}'|$), within both the near and far zones , the first-iterated ${}_1h^{\alpha\beta}$ takes the form of Eq. (2.12). Since $M^{\alpha\beta Q}$ is a function only of $u = t - r$, the spatial gradients ∂_Q produce only unit radial vectors $\hat{n}^i$, powers of r, and retarded time derivatives of $M^{\alpha\beta Q}$. Products of $\hat{n}^i$ can be grouped into symmetric trace-free (STF) products $\hat{n}^{<L>}$, which are analogous to Y_{LM} (see Appendix **??** for useful formulae related to STF products). Thus, outside the source, ${}_1h^{\alpha\beta}$ can be written as a sequence of terms of the form

$${}_1h^{\alpha\beta}{}_{B,L}(t, \mathbf{x}) = f_{B,L}(u)\hat{n}^{<L>}r^{-B} \ . \tag{2.17}$$